Asymptotically one-dimensional diffusions on the Sierpinski gasket and the *abc*-gaskets.

Kumiko Hattori
Department of Pure and Applied Sciences, University of Tokyo,
Meguro-ku, Tokyo 153, Japan
e-mail address: kumiko@tansei.cc.u-tokyo.ac.jp

Tetsuya Hattori
Faculty of Engineering, Utsunomiya University,
Ishii-cho, Utsunomiya 321, Japan
e-mail address: hattori@tansei.cc.u-tokyo.ac.jp

Hiroshi Watanabe
Department of Mathematics, Nippon Medical School,
Nakahara-ku, Kawasaki 211, Japan
e-mail address: watmath@tansei.cc.u-tokyo.ac.jp

November 23, 2004

Abstract

Diffusion processes on the Sierpinski gasket and the *abc*-gaskets are constructed as limits of random walks. In terms of the associated renormalization group, the present method uses the inverse trajectories which converge to unstable fixed points corresponding to the random walks on one-dimensional chains. In particular, non-degenerate fixed points are unnecessary for the construction. A limit theorem related to the discrete-time multi-type non-stationary branching processes is applied.

Mathematics Subject Classification: 60J60, 60J25, 60J85, 60J15

Running title: Asymptotically one-dimensional diffusions on gaskets

1 Introduction.

In this paper we construct what we call an ‘asymptotically one-dimensional’ diffusion process on the Sierpinski gasket.

We begin with some standard notations. For $n \in \mathbf{Z}$ let F_n denote the (infinite) pre-Sierpinski gasket composed of edges of length 2^{-n}. Namely, consider a x_1–x_2 plane and let $O = (0,0)$, $a_0 = (1,0)$, $b_0 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. Let F'_0 be the set of vertices and edges of $\triangle O a_0 b_0$, and F'_n , $n = 0, 1, 2, \cdots$, be a sequence of graphs defined inductively by

$$F'_{n+1} = F'_n \cup (F'_n + 2^n a_0) \cup (F'_n + 2^n b_0) \, , \ n = 0, 1, 2, \dots ,$$

where $A + a = \{x + a \mid x \in A\}$, and $kA = \{kx \mid x \in A\}$. Let F_0 be the union of $\bigcup_{n=0}^{\infty} F'_n$ and its reflection in the x_2-axis. Then $F_n = 2^{-n} F_0$, $n \in \mathbf{Z}$.

Let G_n denote the set of vertices in F_n. Each vertex in F_n, or each element of G_n (including O), has four n-neighbor points; by an n-neighbor point we mean a vertex in F_n joined by an edge of F_n. $G = \overline{\bigcup_{n=0}^{\infty} G_n}$ is the Sierpinski gasket.

For a process X taking values in G we define stopping times $T_{n,i}(X)$, $n \in \mathbf{Z}$, by $T_{n,0}(X) = \inf \{t \geq 0 \mid X(t) \in G_n\}$, and

$$T_{n,i+1}(X) = \inf \{t > T_{n,i}(X) \mid X(t) \in G_n \setminus \{X(T_{n,i}(X))\}\} \ , \ i = 0, 1, 2, \cdots .$$

$T_{n,i}(X)$ is the time that X hits G_n for the $i+1$-th time, counting only once if it hits the same point more than once in a row. Denote by $W_{n,i}(X) = T_{n,i+1}(X) - T_{n,i}(X)$ the time interval to hit two points in G_n.

Next we introduce notions of decimation and renormalization group. For an integer n and a Markov process X on G or on G_N for some $N \geq n$, we call a random walk (Markov chain) X' on G_n defined by $X'(i) = X(T_{n,i}(X))$ an n-decimated walk of X.

Put $\mathbf{R}_+ = \{x \in \mathbf{R} \mid x \geq 0\}$. For $N \in \mathbf{Z}$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbf{R}_+^{\mathbf{3}} \setminus (0,0,0)$, we define a random walk $Z = Z_{N,\alpha}$ on G_N as follows. At each integer time, Z jumps to one of the four N-neighbor points, and the relative rates of the jumps are

(1) α_1, for a jump in horizontal direction (parallel to x_1 axis),

(2) α_2, for a jump in 60° or -120° direction,

(3) α_3, for a jump in 120° or -60° direction.

In the following, we call a jump of the last two types, a diagonal jump. We may choose the parameter space $\mathcal{P}$ for the transition probabilities as

$$\mathcal{P} = \left\{ \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbf{R}_+^{\mathbf{3}} \mid \alpha_1 + \alpha_2 + \alpha_3 = 1 \right\} .$$

Proposition 1.1. *The $(N-1)$-decimated walk of $Z = Z_{N,\alpha}$ with the starting point $Z(0) = x \in G_{N-1}$ has the same law as $Z' = Z_{N-1,T\alpha}$, with the starting point $Z'(0) = x$. Here, $T : \mathcal{P} \to \mathcal{P}$ is a map defined by*

$$T\alpha = (C(\alpha_1 + \alpha_2\alpha_3/3)\,,\ C(\alpha_2 + \alpha_3\alpha_1/3)\,,\ C(\alpha_3 + \alpha_1\alpha_2/3))\,,$$

$$1/C = 1 + (\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1)/3\,.$$

Proof. The essential point is to calculate the transition probability of the $N-1$-decimated walk,

$$\text{Prob}[\ Z(T_{N-1,i+1}(Z)) = y \mid Z(T_{N-1,i}(Z)) = x\ ]\,,$$

which is a little arithmetic as in [7, Sections 4 and 5]. This gives the form of T. (In fact, the quantity above is equal to $g_{x,y}(1,1,1,1)$ of (2.10) with n replaced by $N-1$.) □

We call the dynamical system defined by T on the parameter space $\mathcal{P}$ the renormalization group.

The set of transition probabilities, or the space of random walks, which we consider, is characterized by the 'local translational invariance' (uniformity) and the symmetry of the process. See Proposition A.1 in Appendix A. See [7, Section 5] for our motivation for considering such a space.

We choose a sequence of random walks $Z_N = \{Z_{N,\alpha_N}\}$ in such a way that

$$\alpha_{N-1} = T\alpha_N\,,\ N \in \mathbf{Z}\,, \tag{1.1}$$

and $\lim_{N\to\infty} \alpha_N = (1,0,0) \in \mathcal{P}$. Note that $(1,0,0)$ is an unstable fixed point of T corresponding to the random walk on a one-dimensional chain. This is possible if and only if we put

$$\alpha_N = (\alpha_{N,1}, \alpha_{N,2}, \alpha_{N,3}) \overset{\text{def}}{=} (1+2w_N)^{-1}\ (1, w_N, w_N)\,, \tag{1.2}$$

where, $0 < w_0 < 1$, and w_N, $N = 1,2,3,\cdots$, are defined inductively by

$$w_{N+1} = (6 - w_N)^{-1}\ \left(-2 + 3w_N + \left(4 + 6w_N + 6w_N^2\right)^{1/2}\right)\,,$$

and w_N, $N = -1,-2,-3,\cdots$, are defined inductively by

$$w_{N-1} = \frac{4w_N + 6w_N^2}{3 + 6w_N + w_N^2}\,. \tag{1.3}$$

(One way to see this is to calculate $\alpha_{N-1,2} - \alpha_{N-1,3}$ with (1.1) to see that unless the difference is zero for some N, α_N will not converge to $(1,0,0)$. The rest is easy.) $\{w_N\}$ is strictly decreasing, and $\lim_{N\to\infty} w_N = 0$ and $\lim_{N\to\infty} w_{-N} = 1$. One sees that if $n \leq N$, then Z_n is a n-decimated walk of Z_N. The above equation (1.3) implies that the probabilities of diagonal jumps for Z_N is of order $O(3/4)^N$ (Proposition 2.3 (1)): The finer the structure is, the more the random walker favors horizontal jumps. The decimation covariance (1.1) implies, on the other hand, that in macroscopic scale (i.e. for fixed n) the probabilities of diagonal 'macroscopic' jumps are fixed to some non-zero values. Thus we expect a non-trivial continuum limit process as $N \to \infty$ which has a property that when observed microscopically (short length scale), horizontal moves dominate.

We prove in this paper the following.

Theorem 1.2 (Proposition 3.7, Proposition 3.8, Proposition 3.9). *Let $x \in G$, and let $x_N \in G_N$, $N \in \mathbf{Z}$, be a sequence of points satisfying $\lim_{N\to\infty} x_N = x$. Consider a sequence of random walks Z_N, $N \in \mathbf{Z}$, with transition probability given by (1.2), and with starting point $Z_N(0) = x_N$, $N \in \mathbf{Z}$. Then the sequence of processes X_N, $N \in \mathbf{Z}$, defined by $X_N(t) = Z_N([6^N t])$, converges weakly to a continuous, strong Markov G-valued process X as $N \to \infty$, with $X(0) = x$. The law of X is independent of $\{x_N\}$.*

Denote by P_x the law of X with $X(0) = x$, and let E_x be the expectation with respect to P_x. Define the transition semigroup T_t by $(T_t f)(x) = E_x[f(X(t))]$. Then X is a Feller process, that is T_t maps the space of bounded continuous functions $C_b(G)$ into itself. Furthermore, X is symmetric with respect to the Hausdorff measure μ on G defined by ([13, p.252], [3, p.54])

$$\int f\, d\mu = \frac{2}{3} \lim_{N\to\infty} 3^{-N} \sum_{x \in G_N} f(x)\,.$$

From our standpoint, the previous pioneering works on the construction of diffusion processes on finitely ramified fractals [13, 5, 3, 15, 11, 10] may be regarded as the 'fixed point theories'. Some points in our study compared to the fixed point theories are:

(1) In our work, a set of renormalization group trajectories emerging from the neighborhood of an unstable fixed point is used for construction. Non-degenerate fixed points are unnecessary, in contrast to the fixed point theories.

(2) In our work, a limit theorem related to non-stationary branching processes is used to estimate the number of steps of random walks. The estimate is more subtle than the fixed point theories, where a limit theorem for stationary branching processes was sufficient.

(3) To ensure the continuity of the distributions of the number of steps of random walks, we developed a method applicable to wider class of constructions than the conventional method which uses decimation invariance of the fixed point theories.

The advantage of our method is that we do not need non-degenerate fixed points of the renormalization group for the construction. We will explain a little more on these points.

As is seen from (1.1), the sequence of random walks $\{Z_N\}$ corresponds to the (inverse) trajectory $\{\alpha_N\}$ of the renormalization group T on $\mathcal{P}$. Such a choice is essential to assure decimation covariance Proposition 1.1. In these terms, the works of [13, 5, 3] where the existence of diffusion process on Sierpinski gasket was first established, use the sequence of random walks $Y_N = Z_{N,p_0}$ on G_N, $N \in \mathbf{Z}$, with $p_0 \stackrel{\text{def}}{=} (1/3, 1/3, 1/3) \in \mathcal{P}$. p_0 is a fixed point of the renormalization group T. Therefore the corresponding renormalization group trajectory in $\mathcal{P}$ is a trivial one that stays on the point p_0. Let us call these works the fixed point theories.

In the fixed point theories, the condition that the obtained diffusion process spans the whole fractal implies that the associated random walks have positive transition probabilities to all the neighbor points. This condition is equivalent to the condition that every coordinate component of the fixed point is positive, as in p_0. We call such fixed points non-degenerate fixed points. In the fixed point theories, the existence of non-degenerate fixed points is essential, but their existence is non-trivial. Lindstrøm defined nested fractals in such a way that the renormalization groups have non-degenerate fixed points [15, Sections IV and V]. These works make extensive use of symmetries of fractal figures to assure the existence of non-degenerate fixed points. This causes problem when trying to extend these works to more general geometric objects.

In [7, Section 5.4] we introduced the *abc*-gaskets, which is a generalization of Sierpinski gasket in a way that 111-gasket is the Sierpinski gasket. Taking the parameter space to be $\mathcal{P}$, we showed there that the non-degenerate fixed points of the renormalization groups are absent, if the parameters $a,\ b,\ c$, satisfy certain set of conditions. Thus there are examples of finitely ramified fractals where the fixed point theories cannot be applied. In Theorem 1.2, we consider a set of renormalization group trajectories which 'asymptotically emerge' from a neighborhood of an unstable fixed point. Hence the name 'asymptotically one-dimensional' diffusion. Our method can in principle be appplied to the cases where the renormalization groups have unstable fixed points. In particular, we have checked that our method is applicable to *abc*-gaskets in general. Unstable degenerate fixed points corresponding to the random walk on one-dimensional chains are common in fractals. They also exist in the infinitely ramified fractals such as the Sierpinski carpets and their higher dimensional generalizations. We also emphasize the notion of the renormalization group trajectories, which was implicit in the fixed point theories.

In the case of the fixed point theories on Sierpinski gasket, the fact that p_0 is a fixed point implies that $\{W_{n,i}(Y_N)\}$, $N = n+1, n+2, n+3, \cdots$, is a one-type *stationary* branching process [3, Lemma2.5(b)]. A limit theorem for a stationary branching process then gives a necessary estimate. Situation is similar also for the nested fractals, where a theory of multi-type stationary branching processes can be employed for necessary estimates [11]. To be precise, it is not the theory of branching processes but the recursive structures related to branching processes, that is essential for the construction of diffusions. In any case, it is important to notice that all the constructions are implicitly using (an inverse) trajectory of a renormalization group, which, through decimation covariance, leads to the recursive structures.

We will show in Section 2 (Theorem 2.5) that we can apply a limit theorem of [8] which is related to multi-type *non-stationary* branching processes, to obtain estimates on the asymptotic form of the distribution of $W_{n,i}(Z_N)$, the time interval to hit two points in G_n for the random walk Z_N. Since horizontal jumps and diagonal jumps have different jump probabilities, we must consider more than one variables together (see arguments following (2.2)). Non-stationarity is essential, in that we have to compare the growth rate of the number of steps with the rate of the approach of the starting point of the corresponding trajectory to the fixed point. We have to handle a more singular limit than in the fixed point theories. This results in quantitative differences in the behavior of distributions for horizontal and diagonal jumps in Theorem 2.5 (The condition $2 \notin I(x,y)$ in Theorem 2.5 implies that the bond (x,y) is horizontal).

In Section 2 we also prove the continuity of the limit distribution ($Q_{n,x,y}$ of Theorem 2.8) of the number of steps of the random walks. This is one of a crucial properties when taking the continuum limit of Z_N. In the fixed point theories, the property is obtained through certain functional equations for the characteristic functions of the distributions, which hold only if one is working on the fixed point theories. We therefore developed a method applicable to wider class of constructions than the functional equation method (Lemma 2.7). The random walk representation [7, Section 2] which relates the distribution of number of steps for different scales is the key starting point (Lemma 2.6). In Section 3 we make full use of the estimates in Section 2 to prove the existence of a diffusion process.

We announce here that we have proved that we can apply our methods to *abc*-gaskets of [7]. Therefore we have diffusion processes on any *abc*-gaskets, including those cases where non-degenerate fixed points are absent and hence the fixed point diffusions are absent. This demonstrates how our method may be powerful. We will, however, omit the proof for the *abc*-gaskets here, mainly in order to shorten the paper. The arguments for the *abc*-gaskets in general are the same in idea as the Sierpinski gasket, but rather lengthy in calculation, and we used symbolic manipulation systems on computer to complete the proof in those cases.

We would like to make a couple of remarks. First, let us see how much possible choices of sequences of random walks Z_{N,α_N} , $N = 1, 2, 3, \cdots$, we have for the construction of diffusion processes on the Sierpinski gasket. In our framework this is equivalent to finding an infinite length of (inverse) trajectory,

$\alpha_{N-1} = T\alpha_N$, $N = 1, 2, 3, \cdots$, in the parameter space $\mathcal{P}$. T has one non-degenerate stable fixed point $p_0 = (1/3, 1/3, 1/3)$ and three degenerate unstable fixed points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, corresponding to random walks on chains. One possibility is to stay on the non-degenerate fixed point, which is the choice in [13, 5, 3]. We can show that the only other possibilities to have inifinite (inverse) trajectory of T in $\mathcal{P}$ is the one considered in Theorem 1.2, parametrized by w_0, and its 'rotations', the trajectories approaching $(0, 1, 0)$ or $(0, 0, 1)$ in a similar manner. In this sense our 'asymptotically one-dimensional' diffusions exhaust the possibilities. Note, however, that we have different processes on the Sierpinski gasket if the conditions of symmetry [10] or local translational invariance (uniformity) [9, 12, 16] are violated.

Secondly, we give open questions. Much of the properties of the diffusion processes will be qualitatively similar to those of the fixed point theory. For example, we expect existence of transition density $p_t(x, y)$ as in the case of fixed point diffusion of [3]. However, due to the fact that the transition probabilities are approaching degenerate fixed point, we expect 'log corrections', at least for non-horizontal transition. Also the 'exponents' may have different values.

Since our diffusion is not a fixed point thoery, it will not be scale invariant: For the process X of [13, 5, 3] starting at O, $X(0) = O$, it holds by construction that $2^{-1}X(5t)$ is equal in law to $X(t)$, while for the process X of Theorem 1.2 starting at O, $2^{-1}X(\lambda t)$ will not be equal in law to $X(t)$ for any positive number λ. Then the following two scaling limits become of interest:

(1) (Infra Red scaling limit,) $\lim_{N\to\infty} 2^{-N}X(\lambda^N t)$.

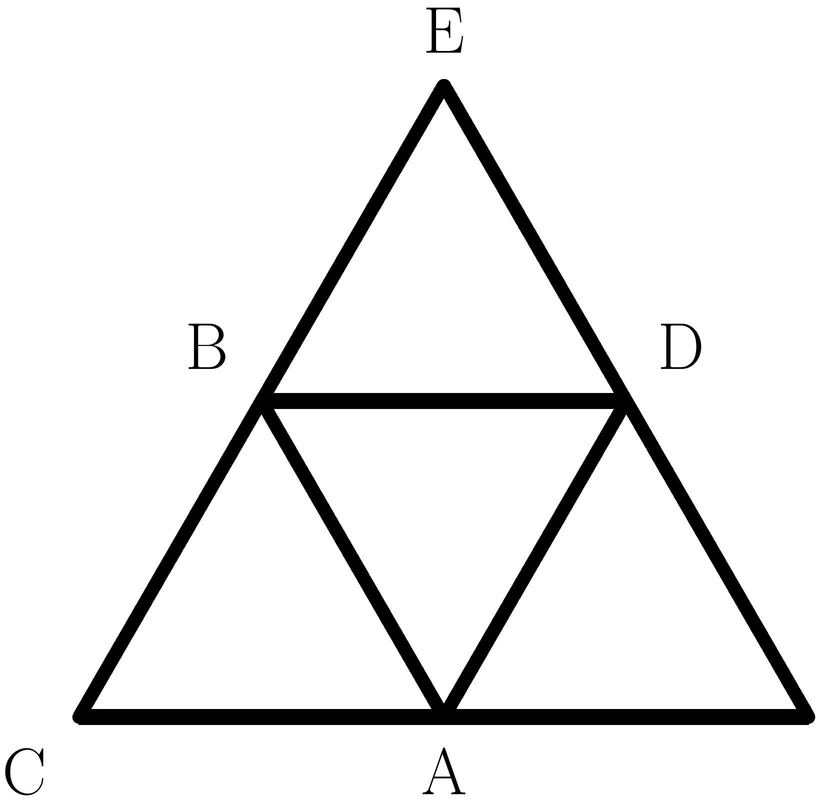
(2) (Ultra Violet scaling limit,) $\lim_{N\to\infty} 2^{N}X(\lambda^{-N} t)$.

From the analysis of the trajectory of the renormalization group, we conjecture that the IR-limit with $\lambda = 5$ will be equal in law to the process of [13, 5, 3]. This suggests an alternate construction method of the diffusion process of their works. The UV limit seems pathological. From Theorem 1.2, the only sensible choice of λ seems to be 6, but UV-scaling limit should be a one-dimensional process, therefore the scaling factor should be 4 if the limit is a Brownian motion. It may be that we do not have a sensible UV limit, but the question is open.

Acknowledgements.
The authors would like to thank Prof. M. T. Barlow, Prof. M. Fukushima, Prof. J. Kigami, Prof. T. Kumagai, Prof. S. Kusuoka, Prof. H. Osada, and Prof. Y. Takahashi, for valuable discussions concerning the present work.

2 Hitting times for the Sierpinski gasket.

Let $n \in \mathbf{Z}$, $N \in \mathbf{Z}$, $N \geq n$, and let Z_N be as in Theorem 1.2. Put $M_{1,n,(i)}^{(N)} \stackrel{\rm def}{=} W_{n,i}(Z_N)$, $i \in \mathbf{Z}_+$. Let $M_{2,n,(i)}^{(N)}$ be the number of diagonal jumps (the jumps


E
B
D
C
A

not parallel to x_1-axis) in the time interval $[T_{n,i}(Z_N)\,,\ T_{n,i+1}(Z_N))$, and let $M^{(N)}_{3,n,(i)}$ be the number of times that the random walk hits those points from which two horizontal lines emerge (those points which are images of $O=(0,0)$ by the local translations of O, in terminology of Appendix A), in the same time interval. Note that for each i, n, and N, $T_{n,i}(Z_N)$ is finite, a.s.. Put $M^{(N)}_{n,(i)} = (M^{(N)}_{1,n,(i)}\,,\ M^{(N)}_{2,n,(i)}\,,\ M^{(N)}_{3,n,(i)})$. By the strong Markov property of the random walk Z_N, we have the following.

Proposition 2.1. *Fix n and N with $N \ge n$, and $x_i \in G_n\,,\ i \in \mathbf{Z}_+$. Then $\left\{M^{(N)}_{n,(i)}\,,\ i \in \mathbf{Z}_+\right\}$ are independent random variables under the conditional probability with conditions $Z_N(T_{n,i}(Z_N)) = x_i\,,\ i \in \mathbf{Z}_+$. If, for some i and j, $x_i = x_j$ and $x_{i+1} = x_{j+1}$, then the laws of $M^{(N)}_{n,(i)}$ and $M^{(N)}_{n,(j)}$ under the conditional probability are the same.*

In view of this Proposition, we shall drop the suffix (i) in the following when no confusion would occur.

Let $(x,y) \in G_n \times G_n$ be an n-neighbor pair. For $m \in \mathbf{Z}^{\mathbf{3}}_+$ put

$$
(2.1) \qquad P^{(N)}_{n,x,y}(m) \stackrel{\text{def}}{=} \text{Prob}[\ M^{(N)}_n = m \mid Z_N(T_{n,i}(Z_N)) = x\,,\ Z_N(T_{n,i+1}(Z_N)) = y\ ]\,,
$$

and define their generating functions by,

$$
(2.2) \qquad f^{(N)}_{n,x,y}(z) \stackrel{\text{def}}{=} \sum_{m \in \mathbf{Z}^{\mathbf{3}}_+} P^{(N)}_{n,x,y}(m)\, z_1^{m_1}\, z_2^{m_2}\, z_3^{m_3}\,, \quad z \in \mathbf{C}^{\mathbf{3}}\,,\ |z_j| \le 1\,,\ j=1,2,3.
$$

Note that Proposition 2.1 implies that $P^{(N)}_{n,x,y}$ and $f^{(N)}_{n,x,y}$ are independent of i.

Next observe that the transition probability of Z_N as given in (1.2) does not discriminate 60° and 120° jumps: $\alpha_{N,2} = \alpha_{N,3}$. Therefore the transition probabilities are invariant under reflection in lines parallel to the x_2-axis. From this symmetry and the local translational invariance defined in Appendix A, we see that each $P^{(N)}_{n,x,y}$ is equal to one of the following 5 cases:

$$
(2.3) \qquad (x,y) = (A,C),\ (A,B),\ (B,D),\ (B,A),\ (B,E),
$$

where, A through E are the 5 vertices of F_n as in Figure 1. As will be seen in the Theorem below, it turns out that $f^{(N)}_{n,x,y}$ are the same for $(x,y) = (B,A)$ and $(x,y) = (B,E)$. Furthermore, due to the consistency condition $f^{(N)}_{n,A,C}(z)\, f^{(N)}_{n,B,A}(z) = f^{(N)}_{n,A,B}(z)\, f^{(N)}_{n,B,D}(z)$, which is also seen to hold below, there are only 3 independent $f^{(N)}_{n,x,y}$'s. In fact this is the reason why we started with 3 component random variables $M_n^{(N)}$. With this remark in mind, define $I(x,y)$ for each pair of n-neighbor points (x,y), by

$$I(x,y) = \begin{cases} \{1,3\}\,, \text{ if } (x,y) \text{ is of type } (A,C), \\ \{1,2,3\}\,, \text{ if } (x,y) \text{ is of type } (A,B), \\ \{1\}\,, \text{ if } (x,y) \text{ is of type } (B,D), \\ \{1,2\}\,, \text{ if } (x,y) \text{ is of type } (B,A) \text{ or } (B,E). \end{cases} \tag{2.4}$$

Theorem 2.2. *For each $n \in \mathbf{Z}$, and $N \in \mathbf{Z}$ satisfying $N \geq n$, and n-neighbor pair (x,y), we have*

$$f_{n,x,y}^{(N)}(z) = \prod_{j \in I(x,y)} z_{n,j}^{(N)}(z)\,. \tag{2.5}$$

Here, $z_n^{(N)}$ are defined recursively by,

$$\begin{aligned} z_n^{(n)}(z) &= z\,, \\ z_n^{(n+1)}(z) &= D_n^{-1}\,F(D_{n+1}\,z)\,, \\ z_n^{(N)}(z) &= z_n^{(n+1)}(z_{n+1}^{(N)}(z))\,,\ \ N > n\,. \end{aligned}$$

D_N are diagonal matrices defined by

$$D_N = \operatorname{diag}(D_{N,1}, D_{N,2}, D_{N,3}) \stackrel{\text{def}}{=} \operatorname{diag}\left(\frac{1}{1+3w_N}\,, w_N\,, \frac{1+3w_N}{2+2w_N}\right), \tag{2.6}$$

with w_N as in (1.2). $F = (F_1, F_2, F_3)$ is a $\mathbf{C}^3$ valued function in three variables defined by:

$$F_1 = y_2/y_4\,, \quad F_2 = y_1/y_2\,, \quad F_3 = y_4/y_3\,, \tag{2.7}$$

with

$$\begin{aligned} y_1(z) &= z_2^2 z_1^2 z_3(1+2z_1z_3)(1+z_1)\,, \\ y_2(z) &= (1-z_1^2)z_1^2z_3^2 + z_2^2z_1^3z_3(1+2z_3+2z_3z_1) + z_2^4z_1^4z_3^2\,, \\ y_3(z) &= (1-z_1^2)(1-2z_1^2z_3^2) - 2z_2^2z_1^2z_3(2+z_1+2z_1z_3+2z_3z_1^2) \\ &\quad + 2z_2^4z_1^4z_3^2\,, \\ y_4(z) &= (1-z_1^2)(1-z_1^2z_3)z_3 - z_2^2z_1^2z_3(3+2z_3+2z_1+4z_1z_3+2z_3z_1^2) \\ &\quad + z_2^4z_1^4z_3^2\,. \end{aligned}$$

Proof. Since $T_{n,i}(Z_n) = i$, we have,

$$M_{1,n,(i)}^{(n)} = W_{n,i}(Z_n) = T_{n,i+1}(Z_n) - T_{n,i}(Z_n) = 1\,.$$

$M_{j,n,(i)}^{(n)}$, $j = 2, 3$, is also counted easily, and with (2.1), we have the statement for $N = n$.

Let $N > n$. Classify each step of a sample path of Z_N into 4 types (A', C'), (A', B'), (B', D'), (B', A'), by the same symmetry consideration as done in (2.3) with similar notations A' through E' for the vertices in F_N as A through E for the vertices in F_n, but with type (B', E') in the same class as (B', A'). For $(x', y') \in \{(A', C'), (A', B'), (B', D'), (B', A')\}$, and for $t \in \mathbf{Z}_+$, denote the number of steps of type (x', y') in the time interval $[T_{n,i}(Z_N)\,,\ T_{n,i}(Z_N) + t)$ by $M'^{(N)}_{n,x',y'}(t)$ (we suppress suffix i, because properties similar to Proposition 2.1 holds for M'). Let $(x, y) \in G_n \times G_n$ be an n-neighbor pair. Define

$$\begin{aligned} P'^{(N)}_{n,x,y}(m') \quad &\overset{\text{def}}{=} \quad \text{Prob}[\ M'^{(N)}_n(W_{n,i}(Z_N)) = m' \mid \\ &\qquad Z_N(T_{n,i}(Z_N)) = x\,,\ Z_N(T_{n,i+1}(Z_N)) = y\ ]\,, \\ &\qquad m' \in \mathbf{Z}^{\mathbf{4}}_+\,, \end{aligned}$$

where

$$M'^{(N)}_n = (M'^{(N)}_{n,A',C'},\ M'^{(N)}_{n,A',B'},\ M'^{(N)}_{n,B',D'},\ M'^{(N)}_{n,B',A'})\,,$$

and define their generating functions by,

$$f'^{(N)}_{n,x,y}(z') \overset{\text{def}}{=} \sum_{m' \in \mathbf{Z}^{\mathbf{4}}_+} P'^{(N)}_{n,x,y}(m') \prod_{j=1}^{4} z'^{m'_j}_j\ ,\ z' \in \mathbf{C}^{\mathbf{4}},\ |z'_j| \le 1,\ j = 1, 2, 3, 4.$$

Proposition 2.1 implies that $P'^{(N)}_{n,x,y}$ and $f'^{(N)}_{n,x,y}$ are independent of i. From Proposition 1.1, (1.1), and the strong Markov property of Z_N, we have

$$f^{(N)}_{n,x,y}(z) = f'^{(n+1)}_{n,x,y}(f^{(N)}_{n+1}(z))\,,\ N > n\,, \tag{2.8}$$

where $f^{(N)}_n = (f^{(N)}_{n,A,C},\ f^{(N)}_{n,A,B},\ f^{(N)}_{n,B,D},\ f^{(N)}_{n,B,A})$. We can show that if $z' = (z_1 z_3,\ z_1 z_2 z_3,\ z_1,\ z_2 z_3)$ for $z = (z_1,\ z_2,\ z_3) \in \mathbf{C}^{\mathbf{3}}$, then

$$f'^{(n+1)}_{n,x,y}(z') = \prod_{j \in I(x,y)} z^{(n+1)}_{n,j}(z)\,. \tag{2.9}$$

We first prove the Theorem by induction in $N - n$, assuming (2.9). The Theorem is already proved for $N - n = 0$, for all $n \in \mathbf{Z}$. Assume that the Theorem holds for an $r = N - n \ge 0$ and for all $n \in \mathbf{Z}$. Then (2.5) holds for $f^{(n+r+1)}_{n+1}(z)$. The formula (2.9) can be applied with z replaced by $z^{(n+r+1)}_{n+1}(z)$, and we see from (2.9) and (2.8) that the Theorem holds for $N - n = r + 1$.

It only remains to prove (2.9). For $x \in G_n$ and $y' \in G_{n+1}$ define

$$\begin{aligned} &g_{x,y'}(z') \overset{\text{def}}{=} \\ &\quad \sum_{m' \in \mathbf{Z}^{\mathbf{4}}_+} \text{Prob}[\ M'^{(n+1)}_n(\sum_{j=1}^{4} m'_j) = m'\,,\ \sum_{j=1}^{4} m'_j \le W_{n,i}(Z_{n+1})\,, \\ &\quad Z_{n+1}(T_{n,i}(Z_{n+1}) + \sum_{j=1}^{4} m'_j) = y' \mid Z_{n+1}(T_{n,i}(Z_{n+1})) = x\ ] \prod_{j=1}^{4} z'^{m'_j}_j\,. \end{aligned}$$

For each $n \in \mathbf{Z}$, $N \in \mathbf{Z}$ with $N \geq n$, and $t \in \mathbf{Z}_+$, we have

$$M'^{(N)}_{n,A',C'}(t) + M'^{(N)}_{n,A',B'}(t) + M'^{(N)}_{n,B',D'}(t) + M'^{(N)}_{n,B',A'}(t) = t \,.$$

Hence for n-neighbor pairs (x, y), the definition of $W_{n,i}$ imply

(2.10) $g_{x,y}(z') =$

$$\sum_{m' \in \mathbf{Z}_+^4} \mathrm{Prob}[\, M'^{(n+1)}_n(W_{n,i}(Z_{n+1})) = m' \,,$$

$$Z_{n+1}(T_{n,i+1}(Z_{n+1})) = y \mid Z_{n+1}(T_{n,i}(Z_{n+1})) = x \,] \prod_{j=1}^{4} z_j'^{m_j'} \,.$$

Therefore

(2.11) $$f'^{(n+1)}_{n,x,y}(z') = g_{x,y}(z')/g_{x,y}(1,1,1,1) \,.$$

Put

$$\begin{aligned} \zeta(z') &= (\zeta_1(z'),\ \zeta_2(z'),\ \zeta_3(z'),\ \zeta_4(z')) \\ &\stackrel{\mathrm{def}}{=} (\frac{z_1'}{2+2w_{n+1}},\ \frac{z_2' w_{n+1}}{2+2w_{n+1}},\ \frac{z_3'}{1+3w_{n+1}},\ \frac{z_4' w_{n+1}}{1+3w_{n+1}}) \,. \end{aligned}$$

The Markov property of Z_{n+1} implies, for $x \in G_n$ and $y' \in G_{n+1}$ in the same triangle (block) of side length 2^{-n}:

$$g_{x,y'}(z') = \sum_{v':n+1\text{-neighbor of } y'} \zeta_{\tilde{I}(v',y')}(z')\, g_{x,v'}(z') + \delta_{x,y'} \,,$$

where $\delta_{x,y'} = 1$ if $x = y'$ and 0 otherwise, and where $\tilde{I}(v', y') = 1, 2, 3, 4$, if (v', y') is of type (A', C'), (A', B'), (B', D'), (B', A'), respectively. Solving these equations for $x = A$, using the symmetry with respect to the reflection in a line parallel to x_2-axis that passes A, and similarly for $x = B$, we obtain, using $z' = (z_1 z_3,\ z_1 z_2 z_3,\ z_1,\ z_2 z_3)$,

$$g_{x,y}(z') = \prod_{j \in I(x,y)} F_j(D_{n+1} z) \,,$$

for each n-neighbor pair (x, y). In particular, using (2.6), (2.7), and (1.3), we have

$$g_{x,y}(1,1,1,1) = \prod_{j \in I(x,y)} D_{n,j} \,,$$

for each n-neighbor pair (x, y). Substituting these results in (2.11) we have (2.9). □

We need some estimates for later use. Put $\vec{1} \stackrel{\rm def}{=} {}^t(1,\cdots,1) \in \mathbf{C^d}$. In the following, $\|\cdot\|$ is the ℓ_2-norm for a vector and the operator norm for a matrix.

Proposition 2.3. *(1) For every $n_0 \in \mathbf{Z}$ there exist positive constants C_1 and C_2 such that w_N in (2.6) satisfies*

$$C_1 \left(\frac{3}{4}\right)^N \le w_N \le \min\left\{C_2 \left(\frac{3}{4}\right)^N, 1\right\}, \ N \ge n_0 \,.$$

In particular, D_N in (2.6) satisfies

$$\sup_{N\ge n_0, i} D_{N,i} < \infty,\ i = 1,2,3\,,$$

and

$$\inf_{N\ge n_0, i} D_{N,i} > 0\,,\ i=1,3\,, \quad \textit{and} \ \inf_{N\ge n_0, i} \left(\frac{4}{3}\right)^N D_{N,2} > 0\,.$$

(2) Let A_N, $N \in \mathbf{Z}$ be 3×3 matrices defined by

$$A_{Nij} = \frac{\partial F_i}{\partial z_j}(D_N\vec{1})\,,$$

with F in (2.7), and let $A = P^{-1}\,\mathrm{diag}\left(6, \frac{8}{3}, \frac{4}{3}\right) P$, with

$$P = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -3 & 6 \\ 1 & -12 & -6 \end{pmatrix}.$$

Then for every $n_0 \in \mathbf{Z}$ there exists a positive constant C_3 such that A_N satisfy

$$\|A_N - A\| \le C_3 \left(\frac{3}{4}\right)^N, \ N \ge n_0\,.$$

(3) For integers n and N satisfying $n \le N$, define $B_{n,N}$ by

$$B_{n,N} \stackrel{\rm def}{=} 6^{-N+n} A_{n+1}A_{n+2}\cdots A_N\,,$$

if $n < N$, and $B_{n,n} \stackrel{\rm def}{=} I$. Then $B_n \stackrel{\rm def}{=} \lim_{N\to\infty} B_{n,N}$ exists, and for every $n_0 \in \mathbf{Z}$ there exists a positive constant C_4 such that

$$\|B_n - B\| \le C_4 \left(\frac{3}{4}\right)^n, \ n \ge n_0,.$$

Here, B is a matrix defined by

$$B = \begin{pmatrix} \frac{6}{7} & \frac{12}{7} & \frac{6}{7} \\ \frac{4}{35} & \frac{8}{35} & \frac{4}{35} \\ \frac{-3}{35} & \frac{-6}{35} & \frac{-3}{35} \end{pmatrix}.$$

$B_{n,N}$ satisfy

$$\|B_{n,N}\| \leq C_5\,, \; N \geq n \geq n_0\,, \tag{2.12}$$

for some positive constant C_5.

(4) *Define $\tilde{F}_N$, $N \in \mathbf{Z}$, by*

$$\tilde{F}_N(t) = F(D_N\vec{1} + t) - D_{N-1}\vec{1} - A_N\, t\,.$$

Then for every $n_0 \in \mathbf{Z}$ there exist positive constants C_6 and C_7 such that

$$\|\tilde{F}_N(t)\| \leq C_6 \left(\frac{4}{3}\right)^N \|t\|^2, \quad \|t\| < C_7 \left(\frac{3}{4}\right)^N, \; N \geq n_0\,.$$

Proof. The first assertion is obvious from the definition (1.3).

Define a $\mathbf{C^3}$-valued function $d(w)$ by

$$d(w) = \left(\frac{1}{1+3w}\,,\; w\,,\; \frac{1+3w}{2+2w}\right), \tag{2.13}$$

and 3×3 matrix $a(w)$ by

$$a(w)_{ij} = \frac{\partial F_i}{\partial z_j}(d(w))\,,\; i = 1,2,3,\; j = 1,2,3. \tag{2.14}$$

Then, by (2.6), we have $A_N = a(w_N)$. From (2.7), $F(z)$ is a rational function in z, hence $a(w)$ is rational in w. By explicit calculations, we see that $a(w)_{ij} = A_{ij} + O(w)$. This with the first assertion imply the second assertion. The third assertion follows from similar arguments as those in [8, Appendix].

To prove the last assertion, put

$$\tilde{F}_w(t) \stackrel{\text{def}}{=} F(d(w) + t) - F(d(w)) - a(d(w))\, t\,,$$

where the matrix a and the vector d are defined in (2.14) and (2.13), respectively. From the first assertion, it is sufficient to prove

$$\|\tilde{F}_w(w\, s)\| \leq C_6\, w \|s\|^2, \quad \|s\| < C_7\,,\; 0 < w \leq 1, \tag{2.15}$$

for some positive constants C_6 and C_7 independent of w and s. There exist polynomials $y_{ip},\ y_{iq},\ i=1,2,3,4$, such that

$$\tilde{y}_i(z) \stackrel{\text{def}}{=} \frac{y_i(z)}{1-z_1} = y_{ip}(z) + \frac{z_2^2}{1-z_1}\, y_{iq}(z)\,,\ i=1,2,3,4. \tag{2.16}$$

Then

$$\begin{aligned}\tilde{y}_i(d(w)+ws) - \tilde{y}_i(d(w)) = \\ y_{ip}(d(w)+ws) - y_{ip}(d(w)) \\ + (1+3w)w\left(\frac{(1+s_2)^2}{3-s_1} y_{iq}(d(w)+ws) - \frac{1}{3} y_{iq}(d(w))\right).\end{aligned} \tag{2.17}$$

Therefore there exists a constant C_8 independent of w and s such that for $i=1,2,3,4$,

$$\|\tilde{y}_i(d(w)+ws) - \tilde{y}_i(d(w))\| < C_8\, w\, \|s\|\,, \quad 0<w\le 1,\ \|s\|<2\,.$$

We then see that there exist positive constants C_9 and $C_{10}<2$ independent of w and s such that

$$\|\tilde{y}_i(d(w)+s)\| > C_9\,, \quad i=2,3,4,\ 0<w\le 1,\ \|s\|<C_{10}\,. \tag{2.18}$$

This with (2.7) and (2.16) implies that the components of $\tilde{F}_w(ws)$ are analytic functions of s for $\|s\|<C_{10}$ whose Taylor expansions around $s=0$ have $O(w)$ coefficients. Therefore (2.15) holds. □

Proposition 2.4. *Let D_N, $N\in\mathbf{Z}$ and F be as in (2.6) and (2.7), respectively.*

(1) For each $n_0\in\mathbf{Z}$ there exist positive constants C_{11} and C_{12} such that for every $n\ge n_0$ and for every pair of n-neighbor points $(x,y)\in G_n\times G_n$, the sequence of generating functions

$$f_{n,x,y}^{(N)}(\exp(-6^{-N}D_N^{-1}s))\,,\ N=n,n+1,\cdots,$$

defined in (2.2), converges uniformly to an analytic function $\psi_{n,x,y}(s)$ as $N\to\infty$ on $\{s\in\mathbf{C}^3 \mid \|s\|<C_{11}\,(9/2)^n\}$. $\psi_{n,x,y}$ satisfies

$$\psi_{n,x,y}(s) = \prod_{j\in I(x,y)} v_{n,j}(s)\,, \tag{2.19}$$

where $v_n(s)=(v_{n,1},\ v_{n,2},\ v_{n,3})$ is an analytic function defined on $\{s\in\mathbf{C}^3 \mid \|s\|<C_{11}\,(9/2)^n\}$ by

$$v_n(s) \stackrel{\text{def}}{=} \lim_{N\to\infty} z_n^{(N)}(\exp(-6^{-N}D_N^{-1}s))\,,$$

with $z_n^{(N)}$ *as in Theorem 2.2.* v_n *has the expression*

$$v_n(s) = \vec{1} - 6^{-n} D_n^{-1} B_n\, s + D_n^{-1} \tilde{v}_n(s)\,,$$

with the bound

$$\|\tilde{v}_n(s)\| \le C_{12} \left(\frac{1}{27}\right)^n \|s\|^2\,, \quad s \in \mathbf{C^3}\,,\ \|s\| < C_{11} \left(\frac{9}{2}\right)^n\,,$$

where B_n *is defined in Proposition 2.3 (3).*

(2) v_n *satisfies*

(2.20) $$v_n(s) = D_n^{-1} F(D_{n+1}\, v_{n+1}(s))\,,\ n \in \mathbf{Z}\,,$$

and B_n *satisfies*

(2.21) $$B_n = 6^{-1} A_{n+1}\, B_{n+1}\,,\ n \in \mathbf{Z}\,.$$

Proof. The first assertion follows from Theorem 2 in [8]. Proposition 2.3 implies that the assumptions for this Theorem are satisfied with $d = 3$, $\ell = 6$, $\delta = 4/3$, D_N and F as in (2.6) and (2.7), respectively. The second assertion is a consequence of the first assertion and the definitions of $z_n^{(N)}$ and B_n in Theorem 2.2 and Proposition 2.3 (3), respectively. □

Proposition 2.4 implies the weak convergence of $6^{-N} W_{n,i}(Z_N)$, the (properly scaled) time interval for Z_N to hit two points in G_n.

Theorem 2.5. *(1) Let* $n \in \mathbf{Z}$*,* $i \in \mathbf{Z}_+$*, and let* $(x, y) \in G_n \times G_n$ *be a pair of* n*-neighbor points. Under the conditional probability with the conditions*

$$Z_N(T_{n,i}(Z_N)) = x\,,\ Z_N(T_{n,i+1}(Z_N)) = y\,, \quad N = n, n+1, n+2, \cdots,$$

the sequence of the random variables $\{6^{-N} W_{n,i}(Z_N)\}$*,* $N = n, n+1, n+2, \cdots$*, converges weakly as* $N \to \infty$*. The limit distribution* $Q_{n,x,y}$ *is independent of* $i \in \mathbf{Z}_+$*, and depends only on* n *and the type of the* n*-neighbor pairs* (x, y)*. It is supported on* $[0, \infty)$ *and satisfies*

$$\int_{\mathbf{R}} \exp(-\omega\, s)\, Q_{n,x,y}(d\omega) = \psi_{n,x,y}(s, 0, 0)\,, \quad |s| < C_{11}\, (9/2)^n\,,\ s \in \mathbf{C}\,,$$

where $\psi_{n,x,y}$ *is as in (2.19).*

(2) Denote by $X_{n,x,y}$ *a random variable whose distribution is equal to* $Q_{n,x,y}$*. Then the mean* $\mathrm{E}[\,X_{n,x,y}\,]$ *satisfies*

$$\mathrm{E}[\,X_{n,x,y}\,] = 6^{-n} \sum_{j \in I(x,y)} D_{n,j}^{-1} B_{n,j,1}\,.$$

Also, there exist $n_1 \in \mathbf{Z}$ and positive constants C_{13} and C_{14}, such that

$$C_{13} \le 6^n \, \mathrm{E}[\, X_{n,x,y} \,] = \sum_{j \in I(x,y)} D_{n,j}^{-1} B_{n,j,1} \,, \tag{2.22}$$

for any (x, y) and $n \ge n_1$, and

$$\sum_{j \in I(x,y)} D_{n,j}^{-1} B_{n,j,1} \le C_{14} \tag{2.23}$$

if $2 \notin I(x, y)$ and $n \ge n_1$.

(3) Put

$$\mathbf{C}_+^{\mathbf{3}} \overset{\text{def}}{=} \{ s \in \mathbf{C}^{\mathbf{3}} \mid \Re(s_i) \ge 0 \,,\ i = 1, 2, 3 \} \,.$$

The function $\psi_{n,x,y}(s)$ in (2.19) is analytically continued to the interior of $\mathbf{C}_+^{\mathbf{3}}$, and is uniquely exetended to a continuous function in $\mathbf{C}_+^{\mathbf{3}}$. Denote the extended function again by $\psi_{n,x,y}(s)$. Then the characteristic function of $X_{n,x,y}$;

$$\begin{aligned} \phi_{n,x,y}(t) &= \mathrm{E}[\, \exp(\sqrt{-1}\, X_{n,x,y}\, t) \,] \\ &= \int_{\mathbf{R}} \exp(\sqrt{-1}\, \omega\, t)\, Q_{n,x,y}(d\omega) \,, \quad t \in \mathbf{R} \,, \end{aligned}$$

satisfies

$$\phi_{n,x,y}(t) = \psi_{n,x,y}(-\sqrt{-1}\, t, 0, 0) \,, \quad t \in \mathbf{R} \,. \tag{2.24}$$

In particular, the real and the imaginary parts of $\phi_{n,x,y}(t)$, $\Re(\phi_{n,x,y}(t))$ and $\Im(\phi_{n,x,y}(t))$ satisfy

$$1 - C_{15}\, 27^{-n}\, t^2 \le \Re(\phi_{n,x,y}(t)) \le 1 \,, \tag{2.25}$$

$$\begin{aligned} & C_{16}\, 6^{-n} t - C_{17}\, 27^{-n}\, t^2 \le \Im(\phi_{n,x,y}(t)) \\ & \quad \le \ C_{18}\, 6^{-n} t + C_{19}\, 27^{-n}\, t^2 \,, \end{aligned} \tag{2.26}$$

$$\textit{for } \ -C_{20} \left(\frac{9}{2}\right)^n < t < C_{20} \left(\frac{9}{2}\right)^n \,, \ n \ge n_1 \,,\ 2 \notin I(x,y) \,,$$

for some positive constants C_{15}, C_{16}, C_{17}, C_{18}, C_{19}, and C_{20}, independent of t, n, and (x, y).

Proof. For each $N \in \mathbf{Z}_+$ satisfying $N \ge n$, let $\tilde{Q}_{n,x,y}^{(N)}$ be the distribution of $6^{-N} D_N^{-1} M_{n,(i)}^{(N)}$ conditioned by $Z_N(T_{n,i}(Z_N)) = x$ and $Z_N(T_{n,i+1}(Z_N)) = y$, and let

$$\begin{aligned} \psi_{n,x,y}^{(N)}(s) &= \int_{\mathbf{R}^{\mathbf{3}}} \exp(-s \cdot w)\, \tilde{Q}_{n,x,y}^{(N)}(dw) \\ &= \int_{[0,\infty)^3} \exp(-s \cdot w)\, \tilde{Q}_{n,x,y}^{(N)}(dw) \,, \quad s \in \mathbf{C}_+^{\mathbf{3}} \,, \end{aligned}$$

be its Laplace transform (generating function). Proposition 2.1 implies that $\tilde{Q}^{(N)}_{n,x,y}$ is independent of $i \in \mathbf{Z}_+$. From (2.1) and (2.2),

$$\psi^{(N)}_{n,x,y}(s) = f^{(N)}_{n,x,y}(\exp(-6^{-N} D_N^{-1} s)) .$$

Proposition 2.4 (1) then implies that $\{\psi^{(N)}_{n,x,y}\}$ converges, as $N \to \infty$, uniformly to the analytic function $\psi_{n,x,y}(s)$ on $\|s\| < C_{11}\,(9/2)^n$. By standard arguments similar to those in [8, Section 1], we see that $\{\tilde{Q}^{(N)}_{n,x,y}\}$ converges weakly to a probability measure as $N \to \infty$. By definition, $W_{n,i}(Z_N) = M^{(N)}_{1,n,(i)}$ and $\lim_{N\to\infty} D_{N,1} = 1$. Hence $\{6^{-N} W_{n,i}(Z_N)\}$ converges weakly as $N \to \infty$. The uniqueness of the extension of $\psi_{n,x,y}$ and (2.24) also follows. Note that Proposition 2.3 (3) implies that for sufficiently large n, $(B_{n,1,1},\, B_{n,2,1},\, B_{n,3,1})$, is close to $(6/7,\ 4/35,\ -3/35)$. Then (2.22) and (2.23) follows from (2.4) and Proposition 2.3 (1). The other statements follow from (2.24), Proposition 2.4 (1), Proposition 2.3 (1), and Proposition 2.3 (3). □

We turn to the random walk representation. For a function F and a positive integer N, let F^N denote the N time iterated function of F. Let $n \in \mathbf{Z}$, and let $N \in \mathbf{Z}$ satisfying $n \le N$. Then Theorem 2.2 and (2.4) imply for any n-neighbor pair $(x, y) \in G_n \times G_n$,

$$f^{(N)}_{n,x,y}(z) = \prod_{j \in I(x,y)} D_{n,j}^{-1} (F^{N-n})_j (D_N z) . \tag{2.27}$$

Let $W^{(N)}_{n,x,y}$ denote the set of walks on G_N from x to y that do not hit points in G_n until they arrive at y:

$$\begin{aligned} W^{(N)}_{n,x,y} = \{ &(b_1, b_2, \cdots, b_\ell) \mid \ell \ge 0 ,\ \text{for } j = 1, 2, 3, \cdots, \ell, \\ &b_j = (x'_{j-1}, x'_j) \in G_N \times G_N \text{ is a pair of } N\text{-neighbor points}, \\ &x'_0 = x\,,\ x'_\ell = y\,,\ x_j \notin G_n\,,\ j = 1, 2, 3, \cdots, \ell - 1 \} . \end{aligned}$$

Lemma 2.6. *Let $n \in \mathbf{Z}$, and let $N \in \mathbf{Z}$ satisfying $n \le N$. For any n-neighbor pair $(x, y) \in G_n \times G_n$, the following holds:*

$$f^{(N)}_{n,x,y}(z) = \sum_{w \in W^{(N)}_{n,x,y}} P(w) \prod_{(x',y') \in w} \left(\prod_{i \in I(x',y')} z_i \right) . \tag{2.28}$$

The convergence of the right hand side is absolute and uniform in $z = (z_1,\ z_2,\ z_3) \in \mathbf{C}^3$, for $|z_i| \le 1$, $i = 1, 2, 3$. $P(w)$ is the probability of the walk w under the conditional probability with conditions $Z_N(T_{n,0}(Z_N)) = x$ and $Z_N(T_{n,1}(Z_N)) =$

y:

$$(2.29)\quad P(w) = \prod_{(x',y')\in w} \mathrm{Prob}[\, Z_N(1) = y' \mid Z_N(0) = x' \,] \times \mathrm{Prob}[\, Z_N(T_{n,1}(Z_N)) = y \mid Z_N(T_{n,0}(Z_N)) = x \,]^{-1}\,.$$

In particular,

$$(2.30)\qquad \phi_{n,x,y}(t) = \sum_{w\in W^{(N)}_{n,x,y}} P(w) \prod_{(x',y')\in w} \phi_{N,x',y'}(t)\,, \quad t\in\mathbf{R}\,.$$

Proof. The arguments for the random walk representation (2.28) are standard [7, Section 2]. From Theorem 2.2, (2.4), and (2.27) we have, for $n \le N \le N+k$,

$$f^{(N+k)}_{n,x,y}(z) = f^{(N)}_{n,x,y}(z^{(N+k)}_N(z))\,.$$

Therefore (2.28), Theorem 2.2, and (2.4) imply

$$f^{(N+k)}_{n,x,y}(z) = \sum_{w\in W^{(N)}_{n,x,y}} P(w) \prod_{(x',y')\in w} f^{(N+k)}_{N,x',y'}(z)\,.$$

Note that for fixed N, n, x, and y, the number of possible (x',y')'s in the right hand side of the above equation is finite and independent of k. Put

$$z = (\exp(-6^{-N-k} D^{-1}_{N+k,1}\sqrt{-1}\,t), 0, 0)\,,\ t\in\mathbf{R}\,,$$

and use Proposition 2.4 (1) and (2.24) to take the limit $k\to\infty$. The uniform convergence of the random walk representation implies (2.30). □

In the next Lemma, C_1, C_2, etc., denote some positive constants independent of k and n.

Lemma 2.7. *Let p_n, q_n, $h^{(k)}_n$, and $g^{(k)}_n$, $n = 0,1,2,\cdots$, $k = 0,1,2,\cdots$, satisfy*

$$(2.31)\qquad C_1\delta^{-n} \le p_n \le C_2\delta^{-n}\,,$$

$$(2.32)\qquad C_3\delta^{-n} \le q_n \le C_4\delta^{-n}\,,$$

for $n = 1,2,\cdots$, and

$$(2.33)\qquad 0 \le h^{(k)}_n \le {h^{(k)}_{n+1}}^{\alpha} + p_{n+1}\,(g^{(k)}_{n+1} - {h^{(k)}_{n+1}}^{\alpha})\,,$$

$$(2.34)\qquad 0 \le g^{(k)}_n \le q_{n+1}\, g^{(k)}_{n+1} + (1-q_{n+1})\, h^{(k)}_{n+1}\,,$$

$$(2.35)\qquad 0 \le h^{(k)}_{n+k} \le 1 - C_5\theta^n\delta^{-k}\,,$$

$$(2.36)\qquad 0 \le g^{(k)}_{n+k} \le 1\,,$$

hold for $n = 0, 1, 2, \cdots$, and $k = 0, 1, 2, \cdots$, where α, δ, and θ, are constants satisfying $\alpha \geq 2$, $1 < \delta < \alpha$, and $0 < \theta < 1$. Then, there exist integers n_0, and $k_0(n)$, $C(n)$, $n \geq n_0$, such that

$$0 \leq h_n^{(k)} \leq \exp(-C_6(k - C(n))^2)\,,\ n > n_0\,,\ k > k_0(n)\,, \tag{2.37}$$

$$0 \leq g_n^{(k)} \leq \exp(-C_7(k - C(n))^2)\,,\ n > n_0\,,\ k > k_0(n)\,. \tag{2.38}$$

Proof. For sufficiently large n, we have $0 < p_{n+m} < 1$ and $0 < q_{n+m} < 1$, hence $h_{n+m}^{(k)} \leq 1$ and $g_{n+m}^{(k)} \leq 1$, for $m = k, k-1, \cdots, 1, 0$.

Define a sequence $\bar{h}_{n+m},\ m = k, k-1, \cdots, 1, 0$, by

$$\begin{aligned} \bar{h}_{n+m} &= \bar{h}_{n+m+1}^{\alpha} + p_{n+m+1}\,(1 - \bar{h}_{n+m+1}^{\alpha})\,,\ m = 0, 1, 2, \cdots, k-1, \\ \bar{h}_{n+k} &= 1 - C_5 \theta^n \delta^{-k}\,. \end{aligned}$$

Obviously, $h_{n+m}^{(k)} \leq \bar{h}_{n+m} \leq 1$, holds. Put $\bar{h}_{n+m} = \exp(-x_{n+m})$. While the condition $0 \leq x_{n+m+1} \leq 1$ is satisfied, we have

$$\alpha x_{n+m+1} \geq x_{n+m} \geq (\alpha - C_8 p_{n+m+1})\, x_{n+m+1}\,.$$

This implies that

$$\frac{1}{2} C_9 \alpha^{k-m} \theta^n \delta^{-k} \leq x_{n+m} \leq C_9 \alpha^{k-m} \theta^n \delta^{-k}\,, \tag{2.39}$$

for a sufficiently large n and for m's such that

$$C_9 \alpha^{k-m-1} \theta^n \delta^{-k} < 1\,.$$

Let

$$k_1 = [(1 - \log_\alpha \delta)k + n \log_\alpha \theta + \log_\alpha C_9]\,, \tag{2.40}$$

where $[x]$ denotes the largest integer bounded by x. Note that $0 < k_1 < k$, if k is sufficiently large for fixed n. Then from (2.39) and (2.40) we have

$$h_{n+k_1}^{(k)} \leq e^{-1/2}\,. \tag{2.41}$$

Define a sequence $\tilde{h}_{n+m},\ m = k_1, k_1 - 1, \cdots, 1, 0$, by

$$\begin{aligned} \tilde{h}_{n+m} &= \tilde{h}_{n+m+1}^2 + C_2 \delta^{-(n+m+1)}\,,\ m = k_1 - 1, \cdots, 2, 1, 0, \\ \tilde{h}_{n+k_1} &= e^{-1/2}\,. \end{aligned}$$

Then, $h_{n+m}^{(k)} \leq \tilde{h}_{n+m}$ holds for $m = k_1, \cdots, 2, 1, 0$. Put

$$\tilde{h}_{n+m} = \exp(-2^{k_1-m-1}) + \delta^{-n-m} r_{n+m}\,,\ m = k_1, \cdots, 2, 1, 0.$$

Then if n is sufficiently large, we have $r_{n+m} \le C_{10}$, $m = k_1, \cdots, 2, 1, 0$. Putting $k_2 = [k_1/2]$, we have, if k is sufficiently large,

$$h^{(k)}_{n+m} \le C_{11}\delta^{-n-m} , \ 0 \le m \le k_2 . \tag{2.42}$$

The assumptions (2.33) and (2.34), together with (2.42) yield

$$h^{(k)}_{n+m} \le C_{11}\delta^{-n-m}(h^{(k)}_{n+m+1} + g^{(k)}_{n+m+1}) , \tag{2.43}$$

$$g^{(k)}_{n+m} \le h^{(k)}_{n+m+1} + C_4\delta^{-n-m}g^{(k)}_{n+m+1} , \tag{2.44}$$

for $m = k_2 - 1, \cdots, 2, 1, 0$. Put

$$R_{n+m} = \max(h^{(k)}_{n+m}, g^{(k)}_{n+m}) .$$

Then, (2.43) and (2.44) implies

$$R_n \le \prod_{0\le 2\mu \le k_2-2} \delta^{-2\mu} \le \delta^{-(k_2-3)^2/4} ,$$

if $k_2 \ge 3$ and n is sufficiently large. □

Theorem 2.8. *For each integer n and each n-neighbor pair (x, y), $Q_{n,x,y}$ is continuous:* Prob$[\ X_{n,x,y} = t\ ] = Q_{n,x,y}\{t\} = 0$, $t \in \mathbf{R}$.

Proof. First we prove that there exists an integer n_0 and a positive constant C_{12} such that

$$\max_{(x,y)\in G_{n+k}\times G_{n+k}:\ \text{horizontal bond}} |\phi_{n+k,x,y}(6^k t)| \le 1 - C_{12}27^{-n}\left(\frac{3}{4}\right)^k , \tag{2.45}$$

$$\left(\frac{3}{4}\right)^k < |t| < 5\left(\frac{3}{4}\right)^k , \ t \in \mathbf{R} , \ n \ge n_0, \ k \in \mathbf{Z}_+ .$$

Let $n \in \mathbf{Z}$, $k \in \mathbf{Z}_+$, and $t \in \mathbf{R}$. Let (x, y) be a pair of $n + k$-neighbor points which, when joined, forms a *horizontal* bond. Denote by x_1 the *diagonal* $n + k + 1$-neighbor of x which is in the $n + k$-triangle containing both x and y. Classify the walks $w \in W^{(n+k+1)}_{n+k,x,y}$ by whether it hits x_1 or not, and define

$$W_1 = \{w \in W^{(n+k+1)}_{n+k,x,y} \mid w \text{ hits } x_1\} .$$

Also denote by y_1 the *horizontal* $n + k + 1$-neighbor of x_1, and define a subset of W_1 by

$$W_2 = \{w \in W_1 \mid w \text{ does not jump to } y_1 \text{ directly after \textit{the first} hit at } x_1\} .$$

Applying (2.30) to $\phi_{n+k,x,y}(6^k t)$, we have

$$
\begin{aligned}
&\phi_{n+k,x,y}(6^k t) \\
&\quad = \sum_{w\in W_1} P(w) \prod_{(x',y')\in w} \phi_{n+k+1,x',y'}(6^k t) \\
&\qquad + \sum_{w\in W^{(n+k+1)}_{n+k,x,y}\setminus W_1} P(w) \prod_{(x',y')\in w} \phi_{n+k+1,x',y'}(6^k t) \\
&\quad = \sum_{w\in W_2} P(w) \prod_{(x',y')\in w} \phi_{n+k+1,x',y'}(6^k t) \\
&\qquad \times \left(1-(1+3w_{n+k+1})^{-2}\phi_{n+k+1,x_1,y_1}(6^k t)\phi_{n+k+1,y_1,x_1}(6^k t)\right)^{-1} \\
&\qquad + \sum_{w\in W^{(n+k+1)}_{n+k,x,y}\setminus W_1} P(w) \prod_{(x',y')\in w} \phi_{n+k+1,x',y'}(6^k t)\,.
\end{aligned}
$$

The factor

$$
\left(1-(1+3w_{n+k+1})^{-2}\phi_{n+k+1,x_1,y_1}(6^k t)\phi_{n+k+1,y_1,x_1}(6^k t)\right)^{-1}
$$

comes from the contribution of walks jumping between x_1 and y_1, to and fro. Put for simplicity of notation, $p = \sum_{w\in W_2} P(w)$, $C' = (1+3w_{n+k+1})^{-2}$, and $\phi = \phi_{n+k+1,x_1,y_1}(6^k t)\phi_{n+k+1,y_1,x_1}(6^k t)$. Then we have

$$
(2.46)\qquad |\phi_{n+k,x,y}(6^k t)| \le 1 - p\, C'^{-1}\left((C'^{-1}-1)^{-1} - |C'^{-1}-\phi|^{-1}\right).
$$

Each walk w in W_2 has at least 2 diagonal jumps; to x_1 and back to the baseline $\overline{xy}$, and the whole walk w is a horizontal jump in $n+k$-scale; the jump from x to y. From Proposition 2.3 (1), (1.2), and (2.29), we therefore see that for any $n_0 \in \mathbf{Z}$ there exists a positive constant C_{13} such that

$$
(2.47)\qquad p > C_{13}\left(\frac{3}{4}\right)^{2(n+k)}, \quad n \ge n_0\,,\ k \in \mathbf{Z}_+\,.
$$

Also for any $n_0 \in \mathbf{Z}$ there exist positive constants C_{14} and C_{15} such that

$$
(2.48)\qquad C_{14}\left(\frac{3}{4}\right)^{n+k} < C'^{-1} - 1 < C_{15}\left(\frac{3}{4}\right)^{n+k}, \quad n \ge n_0\,,\ k \in \mathbf{Z}_+\,.
$$

The estimates (2.48), (2.25), and (2.26) together with $|\phi| \le 1$, imply that there exists a positive constant C_{16} such that

$$
(C'^{-1}-1)^{-1} - |C'^{-1}-\phi|^{-1} > {C_{16}}^2\left(\frac{16}{243}\right)^n\left(\frac{4}{3}\right)^k,
$$

for sufficiently large n, $k \in \mathbf{Z}_+$, $(3/4)^k < |t| < 5\,(3/4)^k$. This with (2.46), (2.47), and (2.48), implies (2.45).

Let C be a positive constant satisfying $1 < C < 5$, and put

$$h_n^{(k)} = \max_{(x,y):\ \text{horizontal bond}} |\phi_{n,x,y}(C\,(9/2)^k)|\,, \tag{2.49}$$

$$g_n^{(k)} = \max_{(x,y):\ \text{diagonal bond}} |\phi_{n,x,y}(C\,(9/2)^k)|\,. \tag{2.50}$$

Then (2.45) implies (2.35) for $n \ge n_0$ and $k \in \mathbf{Z}_+$ with $C_5 = C_{12}$, $\theta = 1/27$, and $\delta = 4/3$. Since $\phi_{n,x,y}$ is a characteristic function, (2.36) also holds.

Define p_{n+1} to be the probability that the random walker Z_{n+1} jumps diagonaly at least once, under the condition that

$$(Z_{n+1}(T_{n,i}(Z_{n+1})),\ Z_{n+1}(T_{n,i+1}(Z_{n+1})) = (x,y)\,,$$

where (x,y) is a horizontal bond which attains maximum in (2.49). Similarly, define q_{n+1} to be the probability that the random walker Z_{n+1} passes only diagonal bonds (i.e. does not jump horizontally) under the condition that

$$(Z_{n+1}(T_{n,i}(Z_{n+1})),\ Z_{n+1}(T_{n,i+1}(Z_{n+1})) = (x,y)\,,$$

where (x,y) is a diagonal bond which attains maximum in (2.50). Then from (2.30) with $N = n+1$ and a graphical consideration, we also see that (2.33) and (2.34) hold with $\alpha = 2$. The estimates (2.31) and (2.32) with $\delta = 4/3$ are obtained by estimating $P(w)$ in (2.30), which is a consequence of (1.2), Proposition 2.3 (1), and graphical considerations. Hence Lemma 2.7 can be applied and we have

$$\begin{aligned} |\phi_{n,x,y}(C\,(9/2)^k)| &\le \exp(-\min\{C_6, C_7\}(k - C(n))^2)\,, \\ & \qquad n > n_0\,,\ k > k_0(n)\,, \end{aligned}$$

for each (x,y). This implies the exponential decay of the characteristic functions $\phi_{n,x,y}$ for all n-neighbor pairs (x,y) and for sufficiently large n. Then (2.33) and (2.34) imply, inductively, the exponential decays of the characteristic functions for all $n \in \mathbf{Z}$, which implies the continuity of the measures $Q_{n,x,y}$. □

3 Convergence of path measures.

Let $G = \overline{\bigcup_{n=0}^{\infty} G_n}$ be the Sierpinski gasket as in Section 1, and $D \stackrel{\text{def}}{=} D([0,\infty); G)$. For $n \in \mathbf{Z}$ and $x \in G_n$ we define a family of probability measures $\{P_x^{(N)}\}$, $N \in \mathbf{Z}_+$, $N \ge n$, on D, as follows:

$$P_x^{(N)}[\ w(0) = x\ ] = 1\,,$$

$$P_x^{(N)}[\ w(t_i) = x_i,\ i = 1, \ldots, r\ ] = \mathrm{Prob}[\ Z_N(x, [6^N t_i]) = x_i,\ i = 1, \ldots, r\ ]\,,$$

where $Z_N(x, \cdot)$ is a random walk on G_N starting from x we studied in Section 2. We use abbreviated notations such as

$$P_x^{(N)}[\ w(0) = x\ ] \stackrel{\mathrm{def}}{=} P_x^{(N)}[\ \{w \in D \mid w(0) = x\}\ ]\,.$$

We write $E_x^{(N)}[\ \cdot\ ]$ for the expectations with respect to $P_x^{(N)}$.

For $w \in D$ and $n \in \mathbf{Z}_+$, let us define

$$T_{n,0}(w) = \inf\{t \ge 0 |\ w(t) \in G_n\}\,,$$

$$T_{n,i+1}(w) = \inf\{t > T_{n,i}(w) \mid w(t) \in G_n \setminus \{w(T_{n,i}(w))\}\}\,,$$

$$W_{n,i} \stackrel{\mathrm{def}}{=} T_{n,i+1}(w) - T_{n,i}(w), i = 0, 1, 2, \cdots.$$

Let $N \ge n$, $x \in G_N$, and $i \in \mathbf{Z}_+$. Assume that (x_j, x_{j+1}), $j = 0, 1, \cdots, i$, are n-neighbor pairs and that $|x_0 - x| < 2^{-n}$. Consider the distribution of $W_{n,j}$, $j = 0, 1, \cdots, i$, under the conditional probability $P_x^{(N)}[\ \cdot \mid w(T_{n,j}(w)) = x_j,\ j = 0, 1, \cdots, i+1\ ]$. As we saw in Section 2, for fixed N and n, the distribution of each $W_{n,i}$, $i = 0, 1, \cdots$, coincides to one of the four distributions corresponding to the types of neighboring points (A, C), (A, B), (B, D), and (B, A). Here we denote these distributions by $Q_{n,I}^{(N)}$, $I = 1, 2, 3, 4$, respectively, and their limit distributions as $N \to \infty$ by $Q_{n,I}$, $I = 1, 2, 3, 4$. In the following, we will often use Theorem 2.5 combined with Theorem 2.8 in the form that

$$\lim_{N \to \infty} Q_{n,I}^{(N)}[\{s | a < s < b\}] = Q_{n,I}[\{s | a < s < b\}], \tag{3.1}$$

for any a and b with $0 \le a < b \le \infty$.

Proposition 3.1. *Let N' and n be non-negative integers satisfying $N' \ge n$, and let $x \in G_{N'}$ and $y \in G_n$ with $|x - y| < 2^{-n}$.*

(1) There exists a positive constant A_1 independent of x, y, n, and N', such that

$$E_x^{(N')}[\ T_{n,0}(w) \mid w(T_{n,0}) = y\ ] \le A_1 \left(\frac{8}{27}\right)^n.$$

(2) The distribution of $T_{n,0}$ under the conditional probability

$$P_x^{(N)}[\ \cdot \mid w(T_{n,0}) = y\ ]\,,$$

converges weakly as $N \to \infty$ to a probability measure on $\mathbf{R}$ with a continuous distribution function.

Proof. We start with defining some functions related to the random walk $Z_m(x')$ starting from $x' \in G_m \setminus G_{m-1}$. Let $M_1(x'), \cdots, M_4(x')$ be the numbers of steps of types (A,C), (A,B), (B,D), and (B,A), respectively, that $Z_m(x')$ takes in the time interval $[0, T_{m-1,0}(Z_m)]$. Let $y' \in G_{m-1}$, $|x'-y'| < 2^{-(m-1)}$. For $z = (z_1, z_2, z_3, z_4) \in \mathbf{C}^4$, define

$$\begin{aligned} & h_{m,x',y'}(z) \\ & \quad \stackrel{\text{def}}{=} \sum_{\vec{n}} \text{Prob}[\ (M_1(x'), \cdots, M_4(x')) = \vec{n} \mid Z_m(x', T_{m-1,0}(Z_m)) = y'\ ] \prod_{i=1}^{4} z_i^{n_i}\,, \end{aligned}$$

where the summation is taken over $\vec{n} = (n_1, n_2, n_3, n_4) \in \mathbf{Z}_+$. For each m, $h_{m,x',y'}$ corresponds to one of the following five functions, according to the relative position of (x', y').

$$h_{m,B,E}(z) = \frac{2w_m+3}{w_m+1} \cdot \frac{w_m(w_m+1)z_4}{g_{m,1}(z)}\,,$$

$$h_{m,A,E}(z) = \frac{2w_m+3}{w_m} \cdot \frac{w_m^2 z_2 z_4}{g_{m,1}(z)}\,,$$

$$\begin{aligned} & h_{m,A,C}(z) \\ & \quad = \frac{2(2w_m+3)}{w_m+3} \times \\ & \qquad \frac{(3w_m+1)^2 z_1 + w_m^2 z_2 z_3 z_4 + w_m^2 (3w_m+1) z_2 z_4 - z_1 z_3^2}{g_{m,2}(z)}\,, \\ & h_{m,B,C}(z) \\ & \quad = \frac{2(3w_m+2)(2w_m+3)}{5w_m^2+11w_m+4} \times \\ & \qquad \frac{2w_m(w_m+1)(3w_m+1)z_4 + w_m z_1 z_3 z_4 + w_m(3w_m+1) z_1 z_4 - w_m^3 z_2 z_4^2}{g_{m,2}(z)}\,, \\ & h_{m,D,C}(z) \\ & \quad = \frac{2(3w_m+2)(2w_m+3)}{(w_m+1)(w_m+4)} \times \\ & \qquad \frac{2w_m(w_m+1) z_3 z_4 + w_m z_1 z_3 z_4 + w_m(3w_m+1) z_1 z_4 + w_m^3 z_2 z_4^2}{g_{m,2}(z)}\,, \end{aligned}$$

where

$$g_{m,1}(z) = 3w_m^2 + 4w_m + 1 - (w_m+1)z_3 - w_m^2 z_2 z_4,$$

and

$$g_{m,2}(z) = 2\{(w_m+1)(3w_m+1)^2 - w_m^2 z_2 z_3 z_4 - w_m^2(3w_m+1) z_2 z_4 - (w_m+1) z_3^2\}.$$

These functions are obtained using the Markov property as in Section 2. They are analytic in $\{z \in \mathbf{C}^4 | |z_i| < 1, i = 1,2,3,4\}$.

From these functions we see that there exist positive constants B_1 and B_2 such that

$$\begin{aligned} \text{(3.2)} \qquad & \mathrm{E}[\, T_{m-1,0}(Z_m(x')) \mid Z_m(x', T_{m-1,0}(Z_m(x'))) = y' \,] \\ &= \frac{d}{dt} h_{m,x',y'}((t,t,t,t))|_{t=1} \\ &\leq B_1 + \frac{B_2}{w_m}, \end{aligned}$$

for all $m \in \mathbf{Z}_+$, $x' \in G_m \setminus G_{m-1}$, and $y' \in G_{m-1}$, satisfying $|x' - y'| < 2^{-(m-1)}$. From the definition of $B_{m,N}$ in Proposition 2.3 (3) together with Theorem 2.2 we have, for an m-neighbor pair (u,v),

$$\begin{aligned} \text{(3.3)} \qquad & \mathrm{E}[\, W_{m,i}(Z_N) \mid Z_N(T_{m,i}) = u, Z_N(T_{m,i+1}) = v \,] \\ &= 6^{N-m} \sum_{j \in I(u,v)} D^{-1}_{m,j} (B_{m,N})_{j,1} D_{N,1}. \end{aligned}$$

Combining (2.6), Proposition 2.3 (1), and (2.12), together with (3.3), we see that

$$\text{(3.4)} \quad 6^{-N}\mathrm{E}[\, W_{m,i}(Z_N) \mid Z_N(T_{m,i}) = u, Z_N(T_{m,i+1}) = v \,] \leq B_3 \left(\frac{2}{9}\right)^m,$$

where B_3 is a positive constant independent of m, N, u' and v'. Applying (3.2) and (3.4) to our process,

$$\text{(3.5)} \qquad E^{(N)}_{x'}[\, T_{m-1,0}(w) \mid w(T_{m-1,0}(w)) = y' \,] \leq B_4 \left(\frac{8}{27}\right)^m,$$

for all $m \in \mathbf{Z}_+$, $N \geq m$, $x' \in G_m \setminus G_{m-1}$, and $y' \in G_{m-1}$, satisfying $|x' - y'| < 2^{-(m-1)}$. The estimate (3.5), combined with the strong Markov property of the random walks, implies for any $x \in G_M$, $y \in G_n$, $N \geq M > n$, and $|x - y| < 2^{-n}$;

$$\begin{aligned} & E^{(N)}_x[\, T_{n,0}(w) \mid w(T_{n,0}(w)) = y \,] \\ &= \sum_{\{y_i\}} \Big(E^{(N)}_x[\, T_{M-1,0}(w) \mid w(T_{M-1,0}) = y_{M-1} \,] \\ &\qquad + E^{(N)}_{y_{M-1}}[\, T_{M-2,0}(w) \mid w(T_{M-2,0}) = y_{M-2} \,] + \cdots \\ &\qquad + E^{(N)}_{y_{n+1}}[\, T_{n,0}(w) \mid w(T_{n,0}) = y \,] \Big) \\ &\qquad \times P^{(N)}_x[\, w(T_{i,0}(w)) = y_i, n+1 \leq i \leq M-1 \mid w(T_{n,0}) = y \,] \\ &\leq A_1 \left(\frac{8}{27}\right)^n, \end{aligned}$$

where the summation is taken over $\{y_i\} = (y_n, y_{n+1}, \cdots, y_M)$ with $y_i \in G_i$, $y_n = y$, $y_M = x$, $|y_i - y_{i-1}| < 2^{-(i-1)}$, $i = n+1, \cdots, M$. Thus we have (1). To prove (2), let us consider the characteristic functions for the hitting times of G_n. First let us define

$$\phi_m^{(N)}(t) = (\phi_{m,A,C}^{(N)}(t), \phi_{m,A,B}^{(N)}(t), \phi_{m,B,D}^{(N)}(t), \phi_{m,B,A}^{(N)}(t)),$$

$$\begin{aligned} &\phi_{m,X,Y}^{(N)}(t) \\ &\overset{\text{def}}{=} \ \mathrm{E}[\ \exp(\sqrt{-1}t6^{-N}W_{m,i}(Z_N) \mid \\ &\qquad (Z_N(T_{m,i}), Z_N(T_{m,i+1})) \text{ are } m\text{-neighbor points of type } (X,Y)\ ], \end{aligned}$$

where $t \in \mathbf{R}$, and $(X,Y) \in \{(A,C),(A,B),(B,D),(B,A)\}$. Then we have,

$$\begin{aligned} &E_x^{(N)}[\ \exp(\sqrt{-1}T_{n,0}(w)t) \mid w(T_{n,0}) = y\ ] \\ &= \sum_{\{y_i\}} \left(\prod_{m=M}^{n+1} h_{m,y_m,y_{m-1}}(\phi_m^{(N)}(t)) \right) \times \\ &\qquad P_x^{(N)}[\ w(T_{i,0}(w)) = y_i, n+1 \le i \le M-1 \mid w(T_{n,0}) = y\ ]. \end{aligned}$$

Since Theorem 2.5 implies that $\phi_m^{(N)}(t)$ converges as $N \to \infty$, the characteristic functions also converge as $N \to \infty$ and we have (2). □

Proposition 3.2. *There is a constant ρ, $0 < \rho < 1$, such that*

$$|P_x^{(N)}[\ w(T_{k,0}(w)) = z\ ] - P_y^{(N)}[\ w(T_{k,0}(w)) = z\ ]| < 2\rho^{m-k-1},$$

for all $N \ge m \ge k \ge 0$, x, $y \in G_N$, $|x-y| < 2^{-m}$, and $z \in G_k$.

Proof. By straightforward calculation we see that for any m and any m-neighbor points (x,y), there exists $z' \in G_{m-1}$ such that

$$P_x^{(N)}[\ w(T_{m-1,0}) = z'\ ] \wedge P_y^{(N)}[\ w(T_{m-1,0}) = z'\ ] \ge \alpha,$$

where α is a positive constant independent of x, y, m, and N. In this particular case, we have $\alpha = \frac{1}{3}$. First, consider the case that x and y are contained in the same m-triangle. By an m-triangle we mean a closed equilateral triangle of side length 2^{-m}, with its vertices and edges lying in F_m. For $n \le m$, denote the three vertices of the n-triangle containing both x and y, by $x_1^{(n)}, x_2^{(n)}, x_3^{(n)}$. The property mentioned above implies that for any $k \le n \le N$, $r = 1,2,3$, any non-negative numbers a_i, b_i, $i = 1,2,3$, and γ, satisfying $\sum_{i=1}^3 a_i = \sum_{i=1}^3 b_i =$

γ, there exist non-negative numbers a'_i and b'_i, $i = 1,2,3$, with $\sum_{i=1}^{3} a'_i = \sum_{i=1}^{3} b'_i = (1-\alpha)\gamma$, such that

$$(3.6) \qquad \Big|\sum_{i=1}^{3} a_i P^{(N)}_{x_i^{(n)}}[\, w(T_{k,0}) = x_r^{(k)}\,] - \sum_{i=1}^{3} b_i P^{(N)}_{x_i^{(n)}}[\, w(T_{k,0}) = x_r^{(k)}\,]\Big|$$

$$= \Big|\sum_{i=1}^{3} a'_i P^{(N)}_{x_i^{(n-1)}}[\, w(T_{k,0}) = x_r^{(k)}\,] - \sum_{i=1}^{3} b'_i P^{(N)}_{x_i^{(n-1)}}[\, w(T_{k,0}) = x_r^{(k)}\,]\Big|\,.$$

Note that there exist non-negative numbers p_i, p'_i, $i = 1,2,3$, satisfying

$$\sum_{i=1}^{3} p_i = \sum_{i=1}^{3} p'_i = 1\,,$$

$$P^{(N)}_x[\, w(T_{k,0}) = x_r^{(k)}\,] = \sum_{i=1}^{3} p_i P^{(N)}_{x_i^{(m)}}[\, w(T_{k,0}) = x_r^{(k)}\,]\,,$$

and

$$P^{(N)}_y[\, w(T_{k,0}) = x_r^{(k)}\,] = \sum_{i=1}^{3} p'_i P^{(N)}_{x_i^{(m)}}[\, w(T_{k,0}) = x_r^{(k)}\,]\,.$$

Using (3.6) iteratively $(m-k-1)$ times, starting with $a_i = p_i$, $b_i = p'_i$, $i = 1,2,3$, we have

$$|P^{(N)}_x[\, w(T_{k,0}) = x_r^{(k)}\,] - P^{(N)}_y[\, w(T_{k,0}) = x_r^{(k)}\,]| < (1-\alpha)^{m-k-1}.$$

If x and y lie in the neighboring m-triangles, then take the common vertex of these two triangles, z and use (3.6) between x and z, and between y and z, respectively. Put $\rho = 1-\alpha$. □

Let $\pi_{[0,M]}$ be the projection of $D([0,\infty);G)$ onto $D([0,M];G)$. $P^{(N)}_{x,M} \stackrel{\text{def}}{=} P^{(N)}_x \circ \pi^{-1}_{[0,M]}$ is a probability measure on $D([0,M];G)$.

In the following, for $x \in G$, assume $\{x_N\}$ is a sequence satisfying $x_N \in G_N$, $N = 1,2,\cdots$, and $\lim_{N\to\infty} x_N = x$.

Proposition 3.3. *$\{P^{(N)}_{x_N,M}\}$, $N = 1,2,\cdots$, is tight for any $M > 0$. If $P_{x,M}$ is the weak limit of a subsequence $\{P^{(N')}_{x_{N'},M}\}$, then*

$$P_{x,M}[\, C([0,M];G)\,] = 1\,.$$

Proof. To prove this proposition, it is sufficient to prove the following [4],

(1) For any $\eta > 0$, there exists an $a > 0$, such that

$$P^{(N)}_{x_N,M}[\,|w(0)| > a\,] \leq \eta\,,$$

for all $N = 1, 2, \cdots$.

(2) For any $\varepsilon > 0$ and $\eta > 0$, there exists a $\delta > 0$ and $N' \in \mathbf{Z}$ such that

$$P^{(N)}_{x_N,M}[\sup_{|s-t|<\delta} |w(s) - w(t)| \geq \varepsilon\,] \leq \eta,$$

for $N \geq N'$.

(1) is obvious. We prove (2) by using (3.1) and the fact that for any k and η, there exist a δ and an N_1 such that

$$P^{(N)}_{x_N}[\,W_{k,i}(w) < \delta\,] < \eta\,,$$

for all $i \in \mathbf{Z}_+$, $N > N_1$ and $x \in G$. Since it goes in a similar way to the proof of Proposition 5.4 in [6], we omit the rest of the proof here. □

In the following part of this section, let $m \in \mathbf{N}$ and assume $h : G^m \longrightarrow \mathbf{C}$ is an arbitrary bounded Lipschitz continuous function. The assumption implies that there are positive constants A and H such that

$$|h(x_1, \cdots, x_m) - h(y_1, \cdots, y_m)| \leq A \max_{1\leq i\leq m} |x_i - y_i|,$$

$$|h(x_1, \cdots, x_m)| \leq H,$$

for $(x_1, \cdots, x_m)$, $(y_1, \cdots, y_m) \in G^m$. Assume also $t_i \in [0, \infty]$, $i = 1, \cdots, m$, $t_1 \leq t_2 \leq \cdots \leq t_m$.

Lemma 3.4. *For any $\varepsilon > 0$, there exists integers k and N_0 with $k < N_0$ such that*

$$\begin{aligned} &|E^{(N)}_{x_N}[\,h(w(t_1), \cdots, w(t_m))\,] \\ &\quad - \sum_{y\in G_k} P^{(N)}_{x_N}[\,w(T^k_0) = y\,]\, E^{(N)}_y[\,h(w(t_1), \cdots, w(t_m))\,]| \\ < &\quad \varepsilon\,, \end{aligned}$$

for all $x \in G$ and $N \geq N_0$.

Proof. For an arbitrary $\varepsilon > 0$, let k_0 be an integer satisfying

$$3 \cdot 2^{-k_0} A < \frac{\varepsilon}{3}\,. \tag{3.7}$$

Let ε_1 be a positive number satisfying,

$$\varepsilon_1 < t_1,$$
$$Q_{k_0,I}[\{s|s<\varepsilon_1\}] < \frac{\varepsilon}{6mH},\ I=1,\cdots,4\,.$$

(3.1) implies there exists an integer N_1 such that

$$Q^{(N)}_{k_0,I}[\{s|s<\varepsilon_1\}] < \frac{\varepsilon}{3mH},\ I=1,\cdots,4\,, \tag{3.8}$$

for $N \geq N_1$. Proposition 3.1 (1) together with Chebyshev's inequality implies that there exist integers k and $N_2 > \max(N_1, k)$ such that

$$P^{(N)}_{x_N}[\ T_{k,0} > \varepsilon_1 \mid w(T_{k,0}) = y\ ] < \frac{\varepsilon}{6H}\,, \tag{3.9}$$

for any $y \in G_k$ satisfying $|x_N - y| < 2^{-k}$, and $N \geq N_2$. From the strong Markov property of the random walks $\{Z_N\}$, we have

$$\begin{aligned}
&E^{(N)}_{x_N}[\ h(w(t_1),\cdots,w(t_m))\ ]\\
&= E^{(N)}_{x_N}[\ h(w(t_1),\cdots,w(t_m)), T_{k,0}\leq\varepsilon_1\ ]\\
&\quad + E^{(N)}_{x_N}[\ h(w(t_1),\cdots,w(t_m)), T_{k,0}>\varepsilon_1\ ]\\
&= \sum_{y\in G_k} P^{(N)}_{x_N}[\ w(T_{k,0})=y\ ]\, P^{(N)}_{x_N}[\ T_{k,0}<\varepsilon_1 \mid w(T_{k,0})=y\ ]\\
&\quad \times E^{(N)}_{x_N}[\ E^{(N)}_y[\ h(w'(t_1-T_{k,0}(w)),\cdots,w'(t_m-T_{k,0}(w)))\ ] \mid\\
&\quad T_{k,0}<\varepsilon_1, w(T_{k,0})=y\ ]\\
&\quad + E^{(N)}_{x_N}[\ h(w(t_1),\cdots,w(t_m)), T_{k,0}>\varepsilon_1\ ]\,,
\end{aligned}$$

where the expectation $E^{(N)}_{x_N}$ is taken over w and $E^{(N)}_y$ is taken over w'. Thus from (3.9), we have

$$\begin{aligned}
&|E^{(N)}_{x_N}[\ h(w(t_1),\cdots,w(t_m))\ ]\\
&\quad - \sum_{y\in G_k} P^{(N)}_{x_N}[\ w(T_{k,0})=y\ ]\\
&\quad \times E^{(N)}_{x_N}[\ E^{(N)}_y[\ h(w'(t_1-T_{k,0}(w)),\cdots,w'(t_m-T_{k,0}(w)))\ ] \mid\\
&\quad T_{k,0}<\varepsilon_1, w(T_{k,0})=y\ ]|\\
&< \frac{\varepsilon}{3},
\end{aligned} \tag{3.10}$$

for $N \geq N_2$. For positive constants $\eta_j < \varepsilon_1$, $j = 1,\cdots,m$, and for $w' \in D$, define

$$r_j \overset{\text{def}}{=} \min\{i\in\mathbf{Z}_+ \mid T_{k_0,i}(w') \geq t_j - \eta_j\},$$

and

$$W_j^* \stackrel{\text{def}}{=} T_{k_0,r_j+1}(w') - T_{k_0,r_j}(w'),\ j = 1,\cdots,m.$$

From (3.8), it follows

$$\begin{aligned}(3.11)\qquad & P_y^{(N)}[\ |w'(t_j) - w'(t_j - \eta_j)| > 3 \cdot 2^{-k_0} \text{ for some } j,\ j = 1,\cdots,m\ ] \\ \le\ & P_y^{(N)}[\ W_j^*(w') < \varepsilon_1 \text{ for some } j,\ j = 1,\cdots,m\ ] \\ \le\ & \frac{\varepsilon}{3H}.\end{aligned}$$

This inequality combined with (3.7) implies that for w with $T_{k,0}(w) \le \varepsilon_1$,

$$\begin{aligned}(3.12)\qquad & |E_y^{(N)}[\ h(w'(t_1 - T_{k,0}(w)),\cdots,w'(t_m - T_{k,0}(w)))\ ] \\ & \quad - E_y^{(N)}[\ h(w'(t_1),\cdots,w'(t_m))\ ]| \\ <\ & \frac{2\varepsilon}{3}.\end{aligned}$$

Noting that $E_y^{(N)}[\ h(w'(t_1),\cdots,w'(t_m))\ ]$ no longer involves w, and combining (3.10) and (3.12), we have the statement. □

Proposition 3.5. *(1) For any $n \in \mathbf{Z}_+$, the sequence*

$$E_x^{(N)}[\ h(w(t_1),\cdots,w(t_m))\ ],\ N = n, n+1,\cdots,$$

converges as $N \to \infty$ uniformly in $x \in G_n$.

(2) For $x \in G$ and $\{x_N\}$ converging to x, $E_{x_N}^{(N)}[\ h(w(t_1),\cdots,w(t_m))\ ]$ converges as $N \to \infty$ and the limit is independent of the choice of $\{x_N\}$. Furthermore, if $\{x_N\}$ satisfies $|x_N - x| < 2^{-N}$, the convergence is uniform in $x \in G$.

Proof. First we will prove (1). For any $\varepsilon > 0$, choose $k \in \mathbf{Z}_+$, $k > n$, satisfying

$$(3.13)\qquad 2^{-k}A < \frac{\varepsilon}{6}.$$

From the strong Markov property of $\{Z_N\}$ and from (3.1), it is possible to choose positive integers L and N_1, independent of $x \in G_n$, such that

$$(3.14)\qquad P_x^{(N)}[\ T_{k,L}(w) < t_m\ ] < \frac{\varepsilon}{6H}$$

for $N > N_1$. For $(x_1,\cdots,x_L) \in (G_k)^L$ and $(\ell_1,\cdots,\ell_m) \in \{0,\cdots,L\}^m$, define a subset of D,

$$B(\{x_i\},\{\ell_j\}) \stackrel{\text{def}}{=} \{w \in D \mid w(T_{k,i}) = x_i,\ i = 1,\cdots,L,\ T_{k,\ell_j} \le t_j < T_{k,\ell_{j+1}},\ j = 1,\cdots,m\}.$$

We have,

$$
\begin{aligned}
&|E_x^{(N)}[\, h(w(t_1),\cdots,w(t_m))\,] - E_x^{(N')}[\, h(w(t_1),\cdots,w(t_m))\,]| \\
&\quad\le\quad |E_x^{(N)}[\, h(w(t_1),\cdots,w(t_m)), T_{k,L}(w) > t_m\,] \\
&\qquad\quad - E_x^{(N')}[\, h(w(t_1),\cdots,w(t_m)), T_{k,L}(w) > t_m\,]| \\
&\qquad\quad + \frac{\varepsilon}{3} \\
&\quad\le\quad |\sum E_x^{(N)}[\, h(w(t_1),\cdots,w(t_m)) \mid B(\{x_i\},\{\ell_j\})\,]\, P_x^{(N)}[\, B(\{x_i\},\{\ell_j\})\,] \\
&\qquad\quad - \sum E_x^{(N')}[\, h(w(t_1),\cdots,w(t_m)) \mid B(\{x_i\},\{\ell_j\})\,]\, P_x^{(N')}[\, B(\{x_i\},\{\ell_j\})\,]| \\
&\qquad\quad + \frac{\varepsilon}{3} \\
&\quad\le\quad |\sum h(x_{\ell_1},\cdots,x_{\ell_m})(P_x^{(N)}[\, B(\{x_i\},\{\ell_j\})\,] - P_x^{(N')}[\, B(\{x_i\},\{\ell_j\})\,])| \\
&\qquad\quad + \frac{2\varepsilon}{3} \\
&\quad\le\quad H \sum P_x^{(N)}[\, w(T_{k,i}(w)) = x_i,\ 1 \le i \le L\,] \\
&\qquad\quad \times |P_x^{(N)}[\, T_{k,\ell_j} \le t_j \le T_{k,\ell_{j+1}},\ 1 \le j \le m \mid w(T_{k,i}(w)) = x_i,\ 1 \le i \le L\,] \\
&\qquad\quad - P_x^{(N')}[\, T_{k,\ell_j} \le t_j \le T_{k,\ell_{j+1}},\ 1 \le j \le m \mid w(T_{k,i}(w)) = x_i,\ 1 \le i \le L\,]| \\
&\qquad\quad + \frac{2\varepsilon}{3} \\
&\quad\le\quad \varepsilon,
\end{aligned}
$$

for large enough N and N', where the summation is taken over $(x_1,\cdots,x_L) \in (G_k)^L$ and $(\ell_1,\cdots,\ell_m) \in \{0,\cdots,L\}^m$, $\ell_1 \le \ell_2 \le \cdots \le \ell_m$. We used (3.13), (3.14) and the fact that

$$
P_x^{(N)}[\, w(T_{k,i}) = x_i,\ i = 1,\cdots,L\,] = P_x^{(N')}[\, w(T_{k,i}) = x_i,\ i = 1,\cdots,L\,]
$$

for $N, N' \ge k$. The last inequality comes from Theorem 2.8. Note that there are only finite kind of the factors $P_x^{(N)}[\, T_{k,\ell_j} \le t_j \le T_{k,\ell_{j+1}},\ 1 \le j \le m \mid w(T_{k,i}(w)) = x_i,\ 1 \le i \le L\,]$ because of the local translational invariance, and they converge as $N \to \infty$.

To prove the assertion (2), note first that by Lemma 3.4 we can take k and N_0, such that

$$
\begin{aligned}
&|E_{x_N}^{(N)}[\, h(w(t_1),\cdots,w(t_m))\,] \\
&\qquad - \sum_{y \in G_k} P_{x_N}^{(N)}[\, w(T_0^k) = y\,]\, E_y^{(N)}[\, h(w(t_1),\cdots,w(t_m))\,]| \\
&\quad < \quad \frac{\varepsilon}{6},\ N \ge N_0\,.
\end{aligned}
$$

Next choose N_1' and N_1 such that $2\rho^{N_1'-k-1} < \dfrac{\varepsilon}{3}$ and $|x_N - x_N'| < 2^{-N_1'}$ for $N > N_1$ and $N' > N_1$, where ρ is as in Proposition 3.2. From these and the decimation covariance of $\{Z_N\}$, we have,

$$|P_{x_N}^{(N)}[\, w(T_0^k) = y \,] - P_{x'_N}^{(N')}[\, w(T_0^k) = y \,]| < \frac{\varepsilon}{3}\,,$$

for $N, N' > N_1$. The assertion (1) implies that we can take N_2 such that

$$|E_y^{(N)}[\, h(w(t_1), \cdots, w(t_m)) \,] - E_y^{(N')}[\, h(w(t_1), \cdots, w(t_m)) \,]| < \frac{\varepsilon}{3}\,,$$

for all $y \in G_k$ and $N, N' > N_2$. Combining these together, we have

$$\begin{aligned}
&|E_{x_N}^{(N)}[\, h(w(t_1), \cdots, w(t_m)) \,] - E_{x_{N'}}^{(N')}[\, h(w(t_1), \cdots, w(t_m)) \,]| \\
\le \;& |\sum_{y \in G_k} \Big(P_{x_N}^{(N)}[\, w(T_0^k) = y \,]\, E_y^{(N)}[\, h(w(t_1), \cdots, w(t_m)) \,] \\
&\quad - P_{x_{N'}}^{(N')}[\, w(T_0^k) = y \,]\, E_y^{(N')}[\, h(w(t_1), \cdots, w(t_m)) \,] \Big) | + \frac{\varepsilon}{3} \\
< \;& \varepsilon,
\end{aligned}$$

for $N, N' \ge \max\{N_0, N_1, N_2\}$. If $\{x_N\}$ satisfies $|x_N - x| < 2^{-N}$, then we can take $N_1 = N'_1 + 1$, from which the uniformity of convergence follows. □

We denote the limit in Proposition 3.5 as $E_x[h(w(t_1), \cdots, w(t_m))]$. The uniformity in convergence stated in Proposition 3.5 leads to the following Proposition.

Proposition 3.6. *$E_x[h(w(t_1), \cdots, w(t_m))]$ is continuous in $x \in G$.*

Proposition 3.7. *If $x_N \in G_N$, $N = 1, 2, \cdots$, $x \in G$ and $x_N \to x$ as $N \to \infty$, then $P_{x_N}^{(N)}$ converges weakly as $N \to \infty$ to a probability measure P_x on $D([0, \infty); G)$ concentrated on $C([0, \infty); G)$.*

Proof. Take

$$h(x_1, \cdots, x_m) = \exp(\sqrt{-1} \sum_{j=1}^{m} s_j x_j)\,, \ (s_1, \cdots, s_m) \in \mathbf{R}^m, \ m = 1, 2, \cdots.$$

The convergence of $E_{x_N}^{(N)}[\, h(w(t_1), \cdots, w(t_m)) \,]$ as $N \to \infty$ together with Proposition 3.3 leads to the weak convergence of $\{P_{x_N, M}^{(N)}\}$, for all $M > 0$, with the limit measure concentrated on $C([0, M]; G)$, and thus $P_{x_N}^{(N)}$ converges weakly as $N \to \infty$ to a probability measure P_x, with $P_x[\, C([0, \infty); G) \,] = 1$. □

Proposition 3.7 implies that $E_x[h(w(t_1), \cdots, w(t_m))]$ actually is the expectation of $h(w(t_1), \cdots, w(t_m))$ under P_x.

Proposition 3.8. *$\{P_x;\ x \in G\}$ is a Feller process.*

Proof. First we show the Markov property. Let $f : G \to \mathbf{C}$ be another bounded Lipschitz continuous function with $|f(x) - f(y)| \le B|x - y|$ and $|f(x)| \le F$ for all $x \in G$ and $y \in G$. From the Markov property of the random walks $\{Z_N\}$, we have

$$(3.15) \qquad \begin{aligned} & E_{x_N}^{(N)}[\, h(w(t_1), \cdots, w(t_m))\, f(w(t_{m+1} + s))\,] \\ &= E_{x_N}^{(N)}[\, h(w(t_1), \cdots, w(t_m)) E_{w(t_{m+1})}^{(N)}[\, f(w'(s))\,]\,]\,. \end{aligned}$$

Using and Proposition 3.7, we see that the left-hand side converges as $N \to \infty$ to

$$E_x[h(w(t_1), \cdots, w(t_m))\, f(w(t_{m+1} + s))]\,.$$

The convergence of the right-hand side to

$$E_x[\, h(w(t_1), \cdots, w(t_m))\, E_{w(t_{m+1})}[f(w'(s))]\,]$$

is obtained from Proposition 3.5, Proposition 3.6, and Proposition 3.7. The Feller property comes from Proposition 3.6. □

Proposition 3.9. *$\{P_x;\ x \in G\}$ is a μ-symmetric process, i.e.*

$$\int (T_t f)(x) g(x)\, d\mu(x) = \int f(x)(T_t g)(x)\, d\mu(x)$$

for any continuous functions f and g with compact support. Here, μ and T_t are as in Theorem 1.2.

Proof. It is sufficient to assume that f and g are Lipfschitz continuous. From the symmetry of the random walk Z_N with respect to μ_N (see the remarks after the proof of Proposition A.1), we have, for any $t \ge 0$,

$$(3.16) \qquad \int E_x^{(N)}[\, f(w(t))\,] g(x)\, d\mu_N(x) = \int f(x) E_x^{(N)}[\, g(w(t))\,]\, d\mu_N(x)\,.$$

Let $x \in G$, and for each $N \in \mathbf{Z}_+$, let x_N be the nearest vertex of G_N. Then Proposition 3.5 implies

$$(T_t f)(x) g(x) = \lim_{N \to \infty} E_{x_N}^{(N)}[\, f(w(t))\,] g(x_N)$$

uniformly in x. This with Proposition 3.6 and Proposition A.2 implies that (3.16) converges to the equation in the statement. □

Appendix

A Random walks and Haussdorff measures.

We first prove (Proposition A.1) that the 'local translational invariance' and the symmetry characterizes the set of transition probabilities which we consider. Then we show (Proposition A.2) that the Hausdorff measure μ of Theorem 1.2 is compatible with our choice of Z_N.

Both of the proofs are easily seen to apply to any *abc*-gaskets, but for notational simplicity we will stick to the Sierpinski gasket.

Fix $N \in \mathbf{Z}$ and consider a pre-Sierpinski gasket F_N. Let $x \in G_N$, and let

$$E: \ (x_1, x_2) \in \mathbf{R^2} \ \to E\,(x_1, x_2) = (x_1 + a, x_2 + b) \in \mathbf{R^2}$$

be a translation operator in Euclid x_1-x_2 plane. We call E a local translation of F_N at x if the following holds:

(1) $E\,x \in G_N$,

(2) $E\,y$ is an N-neighbor of $E\,x$ if and only if y is an N-neighbor of x.

We prove the following.

Proposition A.1. *Let*

$$p_{x,y} = \text{Prob}[\ Z(\ell+1) = y \,|\ Z(\ell) = x\ ]\,,\ x \in G_N\,,\ y \in G_N\,,$$

be a transition probability of a random walk Z on G_N. Suppose it satisfies the following:

(1) (Walk on F_N.) $p_{x,y} = 0$ if x and y are not N-neighbor points.

(2) (Symmetric process.) There exists a measure μ on G_N such that $\mu\,\{x\} > 0\,,\ x \in G_N$, and

$$\mu\,\{x\}\,p_{x,y} = \mu\,\{y\}\,p_{y,x}\,,\ x \in G_N\,,\ \ y \in G_N\,.$$

(3) (Local translational invariance.) For all $x \in G_N$ and for all local translation E of F_N at x, we have $\mu\,\{x\} = \mu\,\{E\,x\}$ and $p_{Ex,Ey} = p_{x,y}$, where $y \in G_N$ is any N-neighbor point of x.

Then the transition probability is in the class considered in Section 1; namely, there exists $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbf{R}^3_+ \setminus (0,0,0)$, such that p_{xy}'s at $x \in G_N$ is non-zero only if y is an N-neighbor of x, and is proportional to:

(1) α_1, if $\overrightarrow{xy}$ is parallel to x_1-axis,

(2) α_2, if $\overrightarrow{xy}$ is in 60° or -120° direction,

(3) α_3, if $\overrightarrow{xy}$ is in 120° or -60° direction.

Proof. There are three types of points in G_N with respect to its neighbor structure:

(1) There are two horizontal edges of F_N attached to the point,

(2) there are two edges in $60°$ directions,

(3) there are two edges in $-60°$ directions.

From the local translational invariance, the set $\{p_{xy} \mid y \in G_N\}$ depends only on the type of the point x. Denote the transition probabilities from the type 1 point to its four neighbors by $p_{1\rightarrow}$, $p_{1\nearrow}$, $p_{1\nwarrow}$, and $p_{1\leftarrow}$, with obvious meaning of notations. Likewise we define $p_{2\swarrow}$, $p_{2\searrow}$, $p_{2\rightarrow}$, $p_{2\nearrow}$, $p_{3\nwarrow}$, $p_{3\leftarrow}$, $p_{3\swarrow}$, $p_{3\searrow}$. Also denote by μ_a the measure $\mu\{x\}$ of the point x of type a, $a = 1, 2, 3$.

Let $x = (0, 0) \in G_N$ and $y = (2^{-N}, 0) \in G_N$. Then x and y are type 1 points and N-neighbors. From symmetry property we have $p_{1\rightarrow} = p_{1\leftarrow}$. Considering other pairs of N-neighbor points, one finds another eleven similar relations, from which it is easy to conclude $p_{2\swarrow} = p_{2\nearrow}$ and $p_{3\searrow} = p_{3\nwarrow}$, and eventually,

$$
\begin{aligned}
\alpha_1 &\stackrel{\text{def}}{=} \mu_1\, p_{1\rightarrow} = \mu_2\, p_{2\rightarrow} = \mu_3\, p_{3\leftarrow}\,, \\
\alpha_2 &\stackrel{\text{def}}{=} \mu_1\, p_{1\nearrow} = \mu_2\, p_{2\nearrow} = \mu_3\, p_{3\swarrow}\,, \\
\alpha_3 &\stackrel{\text{def}}{=} \mu_1\, p_{1\nwarrow} = \mu_2\, p_{2\searrow} = \mu_3\, p_{3\searrow}\,.
\end{aligned}
$$

Then

$$
p_{1\rightarrow} : p_{1\nearrow} : p_{1\nwarrow} = p_{2\rightarrow} : p_{2\nearrow} : p_{2\searrow} = p_{3\leftarrow} : p_{3\swarrow} : p_{3\searrow} = \alpha_1 : \alpha_2 : \alpha_3\,,
$$

together with

$$
\mu_1 : \mu_2 : \mu_3 = 2\alpha_1 + \alpha_2 + \alpha_3 : \alpha_1 + 2\alpha_2 + \alpha_3 : \alpha_1 + \alpha_2 + 2\alpha_3\,.
$$

□

From the above proof it is clear that the random walk Z_N defined in Section 1 is symmetric with respect to a measure μ_N on G_N defined by,

$$
\int_{G_N} f\, d\mu_N = 2^{-1}3^{-N} \sum_{x \in G_N} r_N(x) f(x)\,,
$$

where the weight $r_N(x)$ is

(1) $2\alpha_{N,1} + \alpha_{N,2} + \alpha_{N,3}$, if x is of type 1,

(2) $\alpha_{N,1} + 2\alpha_{N,2} + \alpha_{N,3}$, if x is of type 2,

(3) $\alpha_{N,1} + \alpha_{N,2} + 2\alpha_{N,3}$, if x is of type 3.

The following shows that μ of Theorem 1.2 is compatible with our choice of Z_N.

Proposition A.2. *μ_N converges as $N \to \infty$ to μ of Theorem 1.2 in vague topology, namely, for every continuous function f supported on a compact set of G, $\lim_{N\to\infty} \int f\, d\mu_N = \int f\, d\mu$. (The function f on left hand side is meant to be restricted on G_N.)*

Proof. Consider a subset of G lying inside or on the boundary of an upright triangle of side length 2^{-N} in F_N. The measure μ obviously has a property that the μ-measure of this subset is 3^{-N} [3, Lemma 1.1]. Distribute this mass to the three vertices of the triangle in such a way that the down-left vertex has mass $2^{-1}3^{-N}\alpha_{N,1} + \alpha_{N,2}$, and the down-right vertex has mass $2^{-1}3^{-N}\alpha_{N,3} + \alpha_{N,1}$, and the top vertex has mass $2^{-1}3^{-N}\alpha_{N,2} + \alpha_{N,3}$. Since $\alpha_{N,1} + \alpha_{N,2} + \alpha_{N,3} = 1$ the total mass is unchanged. This replacement reproduces the measure μ_N (embedded in G.) The replacement was moving weights by length of 2^{-N}. Since the total mass of a compact subset of G is finite, we have the statement. □

References

[1] M. T. Barlow, R. Bass, *Construction of Brownian motion on the Sierpinski carpet,* Ann. Inst. Henri Poincaré **25** (1989) 225–257.

[2] M. T. Barlow, R. Bass, *Coupling and Harnack inequalities for Sierpinski carpets,* Preprint.

[3] M. T. Barlow, E. A. Perkins, *Brownian Motion on the Sierpinski Gasket,* Probab. Th. Rel. Fields **79** (1988) 543–623.

[4] P. Billingsley, *Convergence of Probability Measures,* John Wiley & Sons, New York, 1968.

[5] S. Goldstein, *Random walks and diffusion on fractals,* IMA Math. Appl. **8** (1987) 121–129.

[6] K. Hattori, T. Hattori, S. Kusuoka, *Self-avoiding paths on the three dimensional Sierpinski gasket,* Publ. RIMS **29** (1993) 455–509.

[7] K. Hattori, T. Hattori, H. Watanabe, *Gaussian field theories on general networks and the spectral dimensions,* Progr. Theor. Phys. Spplement **92** (1987) 108–143.

[8] T. Hattori, H. Watanabe, *On a limit theorem for non-stationary branching processes,* in E. Cinlar et al. (eds.), *Seminar on Stochastic Processes, 1992,* Birkhauser, Boston, 1993, 173–187.

[9] J. Kigami, *Harmonic calculus on p.c.f. self-similar sets,* Trans. Amer. Math. Soc. **1335** (1993) 721–755.

[10] T. Kumagai, *Construction and some properties of a class of non-symmetric diffusion processes on the Sierpinski gasket,* in K. D. Elworthy, N. Ikeda (eds.), *Asymptotic Problems in Probability Theory: stochastic models and diffusions on fractals,* Longman, Harlow, 1993, 219–247.

[11] T. Kumagai, *Estimates of transition densities for Brownian motion on nested fractals,* Probab. Th. Rel. Fields **96** (1993) 205–224.

[12] T. Kumagai, *Regularity, closedness and spectral dimensions of the Dirichlet forms on p.c.f. self-similar sets,* J. Math. Kyoto Univ **33** (1993) 765–786.

[13] S. Kusuoka, *A diffusion process on a fractal,* Proc. Taniguchi Symp. (1987) 251–274.

[14] S. Kusuoka, X. Y. Zhou, *Dirichlet forms on fractals: Poincaré constant and resistance,* Probab. Th. Rel. Fields **93** (1992) 169–196.

[15] T. Lindstrøm, *Brownian motion on nested fractals,* Mem. Amer. Math. Soc. **420** (1990) 1–128.

[16] H. Osada, *Self-similar diffusions on a class of infinitely ramified fractals,* preprint.

[17] Z. Vondracek, *A characterization of certain diffusions on the Sierpinski gasket by the exit distributions,* preprint.