Open problems to an infinite system of quasi-linear partial differential equations with non-local terms

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1 Main results.

We consider a generalization of a system of first order quasilinear partial differential equations (PDE) in $1+1$ space-time dimensions, such that the method of characteristic curve is effective. We consider non-local (integration) terms and a system of infinitely many components (in fact, equations for a measure valued function), and focus on solutions expressed by certain non-Markovian point process. We first summarize in this section our latest results in [2], and then post open problems in § 2..

1.1 Differential equation with non-local term.

Throughout this paper we fix $T > 0$. Let $W \subset C^1([0,1]\times[0,T];[0,\infty))$ be a set of non-negative valued C^1 functions on $[0,1]\times[0,T]$, and a Borel probability measure λ supported on the Borel measurable space $(W, \mathcal{B}(W))$. $\mathcal{B}(W)$ is the σ-algebra generated by open sets with the topology from the space of continuous functions $C^0([0,1]\times[0,T];[0,\infty)) \supset C^1([0,1]\times[0,T];[0,\infty))$ with the metric given by the supremum norm

(1) $$\|w\|_{\mathrm{T}} = \sup_{(y,t)\in[0,1]\times[0,T]} |w(y,t)|.$$

For the probability space $(W, \mathcal{B}(W), \lambda)$, the Borel sets are defined by the topology of the space of continuous functions $w:\ [0,1]\times[0,T]\to\mathbb{R}$ with maximum norm, and we assume

(2) $$M_W := \int_W \|w\|_{\mathrm{T}}\, \lambda(dw) < \infty, \quad \text{and}$$

(3) $$C_W := \sup_{w\in W} \left\| \frac{\partial w}{\partial y} \right\|_{\mathrm{T}} < \infty.$$

Denote the set of initial ($t=0$) points in the space-time $[0,1]\times[0,T]$, the set of upper stream boundary ($y=0$) points, and their union, the set of initial/boundary points, respectively by

(4) $$\begin{aligned} &\Gamma_b = \{0\}\times[0,T] = \{(0,s) \mid 0 \leqq s \leqq T\}, \\ &\Gamma_i = [0,1]\times\{0\} = \{(z,0) \mid 0 \leqq z \leqq 1\}, \quad \Gamma = \Gamma_b \cup \Gamma_i\,. \end{aligned}$$

For $t \in [0,T]$, denote the set of initial/boundary points up to time t by

(5) $$\Gamma_t = \{(z,s)\in\Gamma \mid t_0 \leqq t\} = \Gamma_i \cup \{(0,t_0)\in\Gamma_b \mid 0 \leqq t_0 \leqq t\},$$

and the set of admissible pairs of the initial/boundary point γ and time t by

(6) $$\Delta_T := \{(\gamma,t)\in\Gamma_T\times[0,T] \mid \gamma\in\Gamma_t\}.$$

Let $\mu_0 = \mu_0(dw\times dz)$ be a Borel probability measure on $(W\times[0,1], \mathcal{B}(W\times[0,1]))$. We assume that $\mu_0 \ll \lambda\times dz$, where dz denotes the standard Lebesgue measure on $[0,1]\subset\mathbb{R}$.

Theorem 1 ([2]) *There exists a unique pair of functions y_C and $\mu_t(dw \times dz)$, where y_C is a function of $(\gamma, t) \in \Delta_T$ taking values in $[0,1]$, and $\mu_t(dw \times dz)$ is a function of $t \in [0,T]$ taking values in the probability measures on $W \times [0,1]$, such that the following hold.*

$y_C((y_0,0),t)$ is non-decreasing in y_0, $y_C((0,t_0),t)$ is non-increasing in t_0, and $y_C(\gamma,t)$ is non-decreasing in t, $y_C(\gamma,t)$ and $\dfrac{\partial y_C}{\partial t}(\gamma,t)$ are continuous, and for each $t \in [0,T]$, $y_C(\cdot,t) : \Gamma_t \to [0,1]$ is surjective. Furthermore, $\int_W h(w)\mu_t(dw \times [y,1))$ is Lipschitz continuous in (y,t), for all bounded measurable $h : W \to \mathbb{R}$, with Lipschitz constant uniform in h satisfying $\sup_{w\in W} |h(w)| \leqq 1$, and finally, the following equation of motion with initial/boundary conditions hold.

$$y_C((y_0,t_0),t_0) = y_0\,, \ (y_0,t_0) \in \Gamma, \tag{7}$$

$$\mu_t(dw \times [0,1)) = \lambda(dw), \quad t \in [0,T], \tag{8}$$

$$\mu_t(W \times [y,1)) = 1 - y, \quad (y,t) \in [0,1] \times [0,T], \tag{9}$$

$$\begin{aligned} &\mu_t(dw \times [y_C((y_0,t_0),t),1)) = \mu_{t_0}(dw \times [y_0,1)) \\ &\quad - \int_{t_0}^{t} \int_{z\in[y_C((y_0,t_0),s),1)} w(z,s)\mu_s(dw \times dz)\,ds, \quad ((y_0,t_0),t) \in \Delta_T\,. \end{aligned} \tag{10}$$

◇

We have stated Lipschitz continuous broad solution in Theorem 1 which is a generalization of Lipschitz solution in [1].

In terms of equation of motion for fluids, (10) is interpreted as the equation of motion of incompressible fluid mixture of infinite components in a line segment, where each component is labelled by its space-time dependent evaporation rate function $w \in W$, and the motion of the fluid mixture is driven by the evaporation. The choice of solution (with initial condition) (9) implies incompressibility of the fluid, so that the fluid is pushed downstream in a way to keep the density of total fluid constant. The boundary condition (8) implies conservation of fluid components, so that the upperstream is filled by the evaporated fluid in such a way that each fluid component is conserved.

Concerning (8), since quantity of fluid components (i.e. the measure λ) is conserved, explicit t dependence is absent if λ is supported on functions constant in t. We will discuss, as an open problem in § 2.2, the existence of solution μ_t which is constant in t, in such cases.

We assumed that μ_0 is absolutely continuous with respect to $\lambda \times dz$. Denote the density function by σ:

$$\mu_0(dw \times dz) = \sigma(w,z)\,\lambda(dw)\,dz, \ (w,z) \in W \times [0,1]. \tag{11}$$

Then (8) and (9) for $t = 0$ respectively implies

$$\int_0^1 \sigma(w,z)\,dz = 1, \ w \in W, \quad \text{and} \tag{12}$$

$$\int_W \sigma(w,y)\,\lambda(dw) = 1, \ y \in [0,1]. \tag{13}$$

Note also that a substitution $y = y_C(\gamma,t)$ in (9) implies $y_C(\gamma,t) = 1 - \mu_t(W \times [y_C(\gamma,t),1))$, with which (10) and (9) imply

$$y_C(\gamma,t) = y_0 + \int_{t_0}^{t} \int_{W\times[y_C(\gamma,s),1)} w(z,s)\mu_s(dw \times dz)\,ds. \tag{14}$$

To keep the integration in (14) finite, the assumption (2) on (W, λ) is a natural limitation. It turned out that to prove Theorem 1 under this weakest condition, it was unavoidable to obtain an explicit formula for the solution given in § 1.3 in terms of a process introduced in § 1.2. In § 2.1 we will consider relaxing the other assumption (3), and raise as an open problem, its effect on uniqueness of the solution.

1.2 Point process with last-arrival-time dependent intensity.

Let $N = N(t)$, $t \geqq 0$, be a non-decreasing, right-continuous, non-negative integer valued stochastic process with $N(0) = 0$, and for each $k \in \mathbb{Z}_+$ define its k-th arrival time τ_k by

$$\tau_k = \inf\{t \geqq 0 \mid N(t) \geqq k\}, \quad k = 1, 2, \ldots, \quad \text{and} \quad \tau_0 = 0. \tag{15}$$

The arrival times τ_k are non-decreasing in k, because N is non-decreasing, and since N is also right-continuous, the arrival times are stopping times; if we denote the associated filtration by $\mathcal{F}_t = \sigma[N(s),\ s \leqq t]$, then $\{\tau_k \leqq t\} \in \mathcal{F}_t$, $t \geqq 0$.

Let ω be a non-negative valued bounded continuous function of (s, t) for $0 \leqq s \leqq t$, and for $k = 1, 2, \ldots$ assume that

$$\mathrm{P}[\, t < \tau_k \mid \mathcal{F}_{\tau_{k-1}} \,] = \exp(-\Omega(\tau_{k-1}, t)) \quad \text{on} \quad t \geqq \tau_{k-1}\,, \tag{16}$$

where, for $t \geqq t_0$ put

$$\Omega(t_0, t) = \int_{t_0}^{t} \omega(t_0, u)\, du. \tag{17}$$

If ω is independent of the first variable, then (16) implies that N is the (inhomogeneous) Poisson process with intensity function ω. We are considering a generalization of the Poisson process such that the intensity function depends on the latest arrival time.

A construction of the point process with last-arrival-time dependent intensity goes as follows. Let $\omega :\ [0,\infty)^2 \to [0,\infty)$ be a non-negative valued bounded continuous function of (s, t) for $0 \leqq s \leqq t$, for which we aim to construct a process satisfying (16). Let ν be a Poisson random measure on $[0,\infty)^2$, with unit constant intensity

$$\mathrm{E}[\, \nu([a, b] \times [c, d]) \,] = (b - a)(d - c), \quad b > a > 0,\ d > c > 0, \quad k \in \mathbb{N}.$$

Define a sequence of hitting times τ_k, $k \in \mathbb{Z}_+$, inductively by $\tau_0 = 0$, and, for $k = 1, 2, \ldots$,

$$\tau_k = \inf\{t \geqq \tau_{k-1} \mid \nu(\{(\xi, u) \in [0,\infty)^2 \mid 0 \leqq \xi \leqq \omega(\tau_{k-1}, u),\ \tau_{k-1} < u \leqq t\}) > 0\,\}. \tag{18}$$

τ_k in (16) is defined by (18), and the process $N(t)$ is defined by the reciprocal relation to (15): $N(t) = \max\{k \in \mathbb{Z}_+ \mid \tau_k \leqq t\}$, $t \geqq 0$. $\{\tau_k \leqq t\}$ is in $\mathcal{F}_t := \sigma[\nu(A);\ A \in \mathcal{B}([0,\infty)^2),\ A \subset [0,\infty) \times [0, t],\ k \in \mathbb{N}]$, and consequently N is adapted to $\{\mathcal{F}_t\}$. Basic formulas, for $N(t)$ to be used in a proof of the main theorem, are in [3].

1.3 Flows and construction of solution.

Define the set of flows Θ_T on $[0, 1] \times [0, T]$ by

$$\begin{aligned} \Theta_T := \{\theta :\ \Delta_T \to [0, 1] \mid\ & \theta((y_0, t_0), t_0) = y_0,\ (y_0, t_0) \in \Gamma_T, \quad \text{continuous,} \\ & \text{surjective and non-increasing in } \gamma \text{ for each } t, \\ & \text{non-decreasing in } t \text{ for each } \gamma\ \}, \end{aligned} \tag{19}$$

where, we define a total order $\succeq$ on the initial/boundary set Γ_T by

$$s \leqq t,\ z \leqq y \ \Leftrightarrow\ (0, T) \succeq (0, t) \succeq (0, s) \succeq (0, 0) \succeq (z, 0) \succeq (y, 0) \succeq (1, 0). \tag{20}$$

Let $\theta \in \Theta_T$. For each $w \in W$ and $z \in [0,1)$ define $\omega = \omega_{\theta,w,z}$, a non-negative valued continuous function of (s,t) satisfying $0 \leqq s \leqq t \leqq T$, by

$$\omega_{\theta,w,z}(s,t) = \begin{cases} w(\theta((z,0),t),t), & \text{if } s=0, \\ w(\theta((0,s),t),t), & \text{if } s>0. \end{cases} \tag{21}$$

Let $\{N_{\theta,w,z} \mid z \in [0,1),\ w \in W\}$ be a set of processes, with each $N_{\theta,w,z}$ being a point process N introduced in § 1.2 with the intensity function in (16) determined by $\omega = \omega_{\theta,w,z}$.

The quantity in (17) for the choice (21) is

$$\Omega_{\theta,w,z}(0,t) = \int_0^t w(\theta((z,0),u),u)\,du, \text{ and } \Omega_{\theta,w}(s,t) = \int_s^t w(\theta((0,s),u),u)\,du. \tag{22}$$

Let μ_0 be as in Theorem 1, and define a function $\varphi_\theta(dw,\gamma,t)$ on $(\gamma,t) \in \Delta_T$ taking values in the measures on W, by

$$\varphi_\theta(dw,\gamma,t) = \int_{z\in[y_0,1)} \mathrm{P}[\, N_{\theta,w,z}(t) = N_{\theta,w,z}(t_0)\,]\,\mu_0(dw \times dz),\ \gamma = (y_0,t_0) \in \Gamma, \tag{23}$$

and define a map $G:\ \Theta_T \to \Theta_T$ by

$$G(\theta)(\gamma,t) = 1 - \varphi_\theta(W,\gamma,t),\ \gamma = (y_0,t_0) \in \Gamma,\ (\gamma,t) \in \Delta_T\,. \tag{24}$$

Theorem 2 ([2]) *The function y_C in Theorem 1 is the unique fixed point of G in (24), and μ_t in Theorem 1 is a measure valued function of $(\gamma,t) \in \Delta_T$ uniquely determined by $\mu_t(dw \times [y_C(\gamma,t),1)) = \varphi_{y_C}(dw,\gamma,t)$, where φ_{y_C} is obtained by the substitution $\theta = y_C$ in (23).* $\diamond$

See [2] for proofs of Theorem 1 and Theorem 2.

Explicit formula of G in (24) is found using the properties of the process introduced in § 1.2;

$$G(\theta)((y_0,0),t) = 1 - \int_{W\times[y_0,1)} e^{-\Omega_{\theta,w,z}(0,t)}\,\sigma(w,z)\,\lambda(dw)\,dz, \tag{25}$$

for $\gamma = (y_0,0) \in \Gamma_i$, and for $\gamma = (0,t_0) \in \Gamma_b \cap \Gamma_t$

$$\begin{aligned} G(\theta)((0,t_0),t) = 1 &- \int_{W\times[0,1)} e^{-\Omega_{\theta,w,z}(0,t)}\,\sigma(w,z)\,\lambda(dw)\,dz \\ &- \int_{W\times[0,1)} \sum_{k\geqq 1} \int_{0\leqq u_1\leqq\cdots\leqq u_k\leqq t_0} w(\theta((z,0),u_1),u_1)\,e^{-\Omega_{\theta,w,z}(0,u_1)} \\ \times \prod_{i=2}^{k} &\left(w(\theta((0,u_{i-1}),u_i),u_i)\,e^{-\Omega_{\theta,w}(u_{i-1},u_i)} \right) \times e^{-\Omega_{\theta,w}(u_k,t)} \prod_{i=1}^{k} du_i\,\sigma(w,z)\,\lambda(dw)\,dz. \end{aligned} \tag{26}$$

2 Open problems.

I came across several questions, while trying to generalize [3] to [2]. I welcome answers!

2.1 Lipschitz continuity condition and uniqueness of solution.

In the beginning we assumed bounded Lipschitz constant condition (3) for the intensity functions, in addition to (2). The latter is a natural condition for (14) to make sense, while the naturality of the former condition (3) is less obvious. We have a following result on this point.

Theorem 3 ([2]) . *Under the condition (2) and*

$$C'_W := \sup_{w \in W} \sup_{(y,t),\,(y',t') \in [0,1]\times[0,T]} |w(y,t) - w(y',t')| < \infty, \tag{27}$$

replacing (3), the map $G:\ \Theta_T \to \Theta_T$ *defined by (25) and (26) has a fixed point.* $\diamond$

The notations in the definition of G are introduced in (13), (12), (4), (5), (6), (19), and (22). Note that $G(\theta) \in \Theta_T$ holds with the assumption (27) which is weaker than (3). It is easy to see that $\Theta_T \subset C^0(\Delta_T; [0,1])$ is a bounded, closed, and convex set. According to the Schauder fixed point theorem, compactness of $G:\ \Theta_T \to \Theta_T$ therefore implies Theorem 3. Since $C^0(\Delta_T;\ [0,1])$ is a bounded set with respect to the supremum norm, the Arzela-Ascoli theorem implies that the following implies compactness of G, under the conditions as in Theorem 3.

Lemma 4 *(i) The map* $G:\ \Theta_T \to \Theta_T$ *is continuous, and*

(ii) the functions in the image set $G(\Theta_T)$ *are equicontinuous.* $\diamond$

A proof of Lemma 4 is in [2, Appendix].

In view of Theorem 3, the assumption (3) would be more crucial for the uniqueness of the solution than for the existence. Considering the fluid picture, breakdown of uniqueness, if it ever does, would likely to occur for an 'unstable' initial condition, that is, the evaporation rate $w(y,t)$ is increasing in y, so that once a small portion of the fluid components begin to move in the downstream direction (larger y), the amount of evaporation increases. (Turning the argument in the other direction, I conjecture that if W contains only those functions $w(y,t)$ which are non-increasing in y, then we have a proof of uniqueness without the assumption (3).) We give a candidate initial condition. We will focus on uniqueness of the fixed point for (25). I have no idea on how to deal with (26).

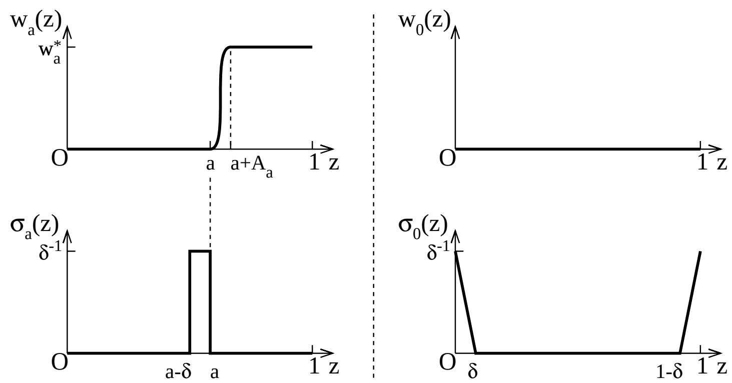

Fig. 1

Fix $\delta \in (0, \frac{1}{2})$, and let w_a^* and A_a be positive valued measurable functions of $a \in [\delta, 1]$. For the probability space $(W, \mathcal{B}(W), \lambda)$ we choose $W = \{w_a \mid a \in \{0\} \cup [\delta, 1]\}$, with

$$w_0(y,t) = 0, \quad \text{and} \quad w_a(y,t) = w_a^*\, f(\frac{y-a}{A_a}),\ \ \delta \leqq a \leqq 1, \tag{28}$$

and

$$\lambda(dw_a) = \mathbf{1}_{[\delta,1]}\, da + \delta\, \delta_0(da),\ \ a \in \{0\} \cup [\delta, 1]\}, \tag{29}$$

where δ_0 is the unit measure concentrated on 0, and $f:\ \mathbb{R} \to [0,1]$ is a C^1 function satisfying

$$f(z) = 0,\ z \leqq 0, \quad 0 < f(z) < 1,\ 0 < z < 1, \quad f(z) = 1,\ z \geqq 1, \tag{30}$$

and we performed a change in the integration variable $w \mapsto a$ by $w = w_a$. We choose the density function $\sigma_a(y) = \sigma(w_a, y)$ of the initial data defined by $\sigma_a(y)\,dy\,\lambda(dw_a) = \mu_0(da \times dy)$, as

$$\sigma_a(y) = \begin{cases} \dfrac{1}{\delta}\phi(\dfrac{1}{\delta}(y-a+\delta)), & \delta \leqq a \leqq 1, \\ \dfrac{1}{\delta^2}(\delta - y)_+ + \dfrac{1}{\delta^2}(y-(1-\delta))_+, & a = 0, \end{cases} \tag{31}$$

where $\phi(z) = \begin{cases} 1, & 0 \leqq z < 1, \\ 0, & z < 0 \text{ or } z \geqq 1, \end{cases}$ and we used a notation $x_+ := \dfrac{1}{2}(x+|x|)$. The assumptions (13), (12), (2), and $\lambda(W) = 1$ hold, if

$$M_w := \int_\delta^1 w_a^*\,da < \infty. \tag{32}$$

Substituting (29) and (31) in (25), and using (22), we have for $0 \leqq y_0 < 1$ and $0 \leqq t \leqq T$,

$$\begin{aligned} & y_C((y_0,0),t) = G(y_C)((y_0,0),t) := I_1(y_0,t) - I_2(y_C)(y_0,t), \quad \text{where} \\ & I_1(y_0,t) = \begin{cases} 1 - \delta + y_0 - \dfrac{1}{2\delta}y_0^2, & 0 \leqq y_0 < \delta, \\ \dfrac{1}{2}, & \delta \leqq y_0 \leqq 1-\delta, \\ y_0 + \dfrac{1}{2\delta}(1-y_0)^2, & 1-\delta < y_0 < 1, \end{cases} \\ & I_2(y_C)(y_0,t) = \int_{y_0}^1 dz \int_{z\vee\delta}^{1\wedge(z+\delta)} da\, e^{-w_a^* \int_0^t f(\frac{1}{A_a}(y_C((z,0),u)-a)\,du}\,dz. \end{aligned} \tag{33}$$

It follows from $w_a(z)\sigma_a(z) = 0$ for all z and a that there is a constant flow $y_C((y_0,0),t) = y_0$ for all y_0 and t. The problem is to decide if there are $\delta \in (0, \frac{1}{2})$, A_a, w_a^*, and f, satisfying (32) and (30) such that (33) holds for a non-constant flow, continuously increasing in t and y_0.

Suppose we are on the side that such fixed point y_C exists, how could we prove it? A standard way is to apply the Schauder's fixed point theorem as in Theorem 3, with set of functions (domain of G) Θ_T replaced by $\tilde{\Theta}_T$ which excludes the constant flow. A candidate set might perhaps look like

$$\tilde{\Theta}_T := \{\theta :\in \Theta_T \mid \theta(y,t) \geqq y + \epsilon\, t,\ t \leqq \epsilon(1-y)^M,\ 0 \leqq y < 1\}, \tag{34}$$

for some $\epsilon > 0$ and $M > 0$. The set (34) excludes the constant flow, is a bounded, closed, and convex set. Since we are restricting the domain of G, the compactness of the map G stated in Lemma 4 is preserved, if (27) holds. This requires boundedness of w_a^*, so we may as well put $w_a^* = 1,\ \delta \leqq a \leqq 1$. If this works, it only remains to prove, for some set of δ, A_a, f, ϵ, M,

$$G(\tilde{\Theta}_T) \subset \tilde{\Theta}_T \quad (?). \tag{35}$$

2.2 Stationary solution.

The boundary condition (8) is designed as a necessary condition for (10) to have a stationary solution for $\mu_t(dw,[y,1))$, in the case when W is supported on intensity functions w which are constant in t. Here, a stationary solution means a solution satisfying $\dfrac{\partial \mu_t}{\partial t} = 0,\ t \in [0,T]$, in (10). Dropping t from the notations $w(y,t)$ and replacing μ_t by μ_0 and using (11), we have

$$\int_{y_0}^{y_C((y_0,t_0),t)} \sigma(w,z)\,\lambda(dw)\,dz = \int_{t_0}^t \int_{z\in[y_C((y_0,t_0),s),1)} w(z)\sigma(w,z)\,\lambda(dw)\,dz\,ds, \quad w \in W.$$

Differentiating by t, and using (14) with (11), and finally replacing $y_C((y_0,t_0),t)$ by y, we have

$$(36)\quad \Big(\int_{W\times[y,1)} w(z)\sigma(w,z)\,\lambda(dw)\,dz\Big)\,\sigma(w,y) = \int_y^1 w(z)\sigma(w,z)\,dz, \quad (w,y)\in W\times[0,1).$$

The question is to decide whether (36) has a solution $\sigma:\ W\times[0,1)\to[0,\infty)$ which is measurable and satisfies (13) and (12). For simplicity, we assume in the following

$$(37)\qquad \inf_{y\in[0,1]} w(y) > 0, \quad w\in W.$$

We can reduce the unknown from a function σ on $W\times[0,1]$ to that on $[0,1]$, as follows. Put

$$(38)\qquad \tilde u(w,y) = \int_y^1 w(z)\,\sigma(w,z)\,dz \quad\text{and}\quad v(y) = \int_W \tilde u(w,y)\lambda(dw).$$

Then (36) implies $\dfrac{\tilde u'(w,y)}{\tilde u(w,y)} = -\dfrac{w(y)}{v(y)}$ which further implies

$$(39)\qquad \tilde u(w,y) = C(w)\,e^{-\int_0^y \frac{w(z)}{v(z)}\,dz},$$

for some function $C:\ W\to[0,\infty)$. With (38) we further have

$$(40)\qquad v(y) = \int_W C(w)\,e^{-\int_0^y \frac{w(z)}{v(z)}\,dz}\,\lambda(dw).$$

A boundary condition $\tilde u(w,1)=0$ is implied from (38), which, with (37), further implies

$$(41)\qquad v(1-) = 0.$$

Note that (38), (39), and (40) imply $\int_W \sigma(w,y)\,d\lambda(dw) = 1$, so that (13) holds. On the other hand, (12) implies with a similar argument,

$$(42)\qquad C(w) = \Big(\int_0^1 \frac{1}{v(z)} e^{-\int_0^z \frac{w(x)}{v(x)}\,dx}\,dz\Big)^{-1}.$$

This and (40) imply

$$(43)\qquad v(y) = H(v)(y) := \int_W \frac{e^{-\int_0^y \frac{w(z)}{v(z)}\,dz}\,\lambda(dw)}{\int_0^1 \frac{1}{v(z)} e^{-\int_0^z \frac{w(x)}{v(x)}\,dx}\,dz}, \quad y\in[0,1).$$

We will define $H(v)(1) := 0$ to save any further remarks on compactness issues at $y=1$. If we have a non-increasing, continuous, non-negative function $v:\ [0,1)\to[0,\infty)$ which satisfies (43) and (41), then (43) and (42) determine (39) and σ is given by differentiating (38), which satisfies (36), (13), and (12).

Note that (42), (1), and (37) imply $\dfrac{1}{C(w)} = \displaystyle\int_0^1 \frac{1}{v(z)} e^{-\int_0^z \frac{w(x)}{v(x)}\,dx}\,dz \geqq \frac{1}{\|w\|_{\mathrm{T}}}$. This with (43) and (2) further imply

$$(44)\qquad H(v)(y) \leqq M_W < \infty.$$

Let D be the set of non-increasing, continuous functions $f: [0,1] \to [0, M_W]$, satisfying $f(1) = 0$. Then D is bounded, closed, and convex. Moreover, (43) and (44) imply $H(D) \subset D$. Therefore, if we can prove the compactness of the map H, namely, the continuity of the map H and the equicontinuity of the functions in the image set $H(D)$, the Schauder's fixed point theorem implies existence of v, and consequently, of a stationary solution to the original PDE.

If W is supported on constant w's, then there is a stationary solution to the original problem. For $w \in W$ let $w^* \in [0,\infty) \setminus \{0\}$ be its (constant) value. Then (42) and (41) imply

$$\frac{1}{C^*(w^*)} := \frac{1}{C(w)} = -\frac{1}{w^*}\int_0^1 \frac{d}{dz}\biggl(e^{-w^* \int_0^z \frac{1}{v(x)}\,dx}\biggr) dz = \frac{1}{w^*}\,, \tag{45}$$

and (40) further implies, with $\lambda^*(dw^*) = \lambda(dw)$ the image measure of λ, supported on $[0,\infty)$,

$$v(y) = -v(y)\,\frac{d}{dy}\biggl(\int_0^\infty e^{-w^* \int_0^y \frac{dz}{v(z)}} \lambda^*(dw^*)\biggr).$$

Dividing by $v(y)$ and integrating, we have a formula which is expressed as

$$\int_0^y \frac{dz}{v(z)} = \varphi^{-1}(1-y), \tag{46}$$

where $\xi = \varphi^{-1}(x)$ is the inverse function of $\varphi(\xi) = \displaystyle\int_0^\infty e^{-w^*\xi} d\lambda^*(dw^*)$, $\xi \geqq 0$.

Since we assume (37) in this subsection, we have $\varphi(0) = 1$ and $\varphi(\infty) = 0$, so that

$$\varphi^{-1}(0) = \infty \ \text{ and } \ \varphi^{-1}(1) = 0. \tag{47}$$

(41) and (37) imply that (47) is consistent with (46). Substituting (46) and (45) in (39), we have $\tilde{u}^*(w^*, y) := \tilde{u}(w, y) = w^*\, e^{-w^*\,\varphi^{-1}(1-y)}$. Substituting this in (38), we arrive at $\sigma^*(w^*, y) := \sigma(w,y) = -\dfrac{d}{dy}\Bigl(e^{-w^*\,\varphi^{-1}(1-y)}\Bigr)$ and $\mu(dw \times [y,1)) = \lambda(dw) \displaystyle\int_y^1 \sigma(w,z)\,dz = e^{-w^*\,\varphi^{-1}(1-y)}\,\lambda(dw)$. Thus for the constant w case, we have an explicit formula for the solution μ in terms of the generating function of λ.

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