Point process with last-arrival-time dependent intensity and 1-dimensional incompressible fluid system with evaporation.

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Abstract

We consider an infinite system of quasilinear first-order partial differential equations, generalized to contain spatial integration, which describes an incompressible fluid mixture of infinite components in a line segment whose motion is driven by unbounded and space-time dependent evaporation rates. We prove unique existence of the solution to the initial-boundary value problem, with conservation-of-fluid condition at the boundary. The proof uses a map on the space of collection of characteristics, and a representation based on a non-Markovian point process with last-arrival-time dependent intensity.

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1 Introduction.

Consider an incompressible fluid mixture in a line segment, say [0,1], which flow in order preserving manner and in one direction, with $y=0$ being the upper stream boundary, and no leaking occurs at $y=1$. Each fluid component, say $\alpha$, evaporates with rate $w_\alpha$ which may vary among different components and may depend on time. Flow of the fluid is driven by filling the evaporated portion of the fluid toward the down stream. To formulate a system of partial differential equations which explains the dynamics of this fluid up to time $T>0$, let $U_\alpha(y,t)$ be the total volume (length) of fluid component $\alpha$ at time $t$ in the interval $[y,1)$. Then we have

\[
\frac{\partial U_\alpha}{\partial t}(y,t) + \sum_\beta w_\beta(t)U_\beta(y,t)\frac{\partial U_\alpha}{\partial y}(y,t) = -w_\alpha(t)U_\alpha(y,t), \quad (y,t) \in [0,1] \times [0,T].
\]

We will preserve the total volume of each component by supplying the evaporated portion from upper stream boundary through the boundary condition

\[
U_\alpha(0,t) = r_\alpha \quad \text{and} \quad U_\alpha(1,t) = 0, \quad t \geq 0,
\]

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for non-negative constants $r_{\alpha}$ satisfying $\sum_{\alpha} r_{\alpha} = 1$. The incompressibility condition is formulated as
\begin{equation}
\sum_{\alpha} U_{\alpha}(y, t) = 1 - y, \quad t \geq 0.
\end{equation}
The number of fluid components may be finite or infinite. (For the latter case we regard the summations in $\alpha$ as series.) If the system is infinite, we should impose an additional condition
\begin{equation}
\sum_{\alpha} r_{\alpha} \sup_t w_{\alpha}(t) < \infty,
\end{equation}
to keep the velocity of the flow finite, namely, to keep coefficient of the $y$-derivative term in (1) well-defined at $y = 0$. With appropriate initial conditions, these define an initial/boundary value problem of a one dimensional first order quasilinear partial differential equations.

In this paper we consider a generalization of (1) to allow for spacial dependence for the evaporation rates $w_{\alpha}$, as well as time dependence. Such generalization seems practically natural, because if the fluid container has spacial non-uniformity in temperature, the evaporation rates would also have spacial dependence. $U_{\alpha}(y, t)$ is the volume of type $\alpha$ fluid component in the interval $[y, 1)$, and we need to consider its density to consider spacially varying evaporation rates, hence a natural generalization of (1) is
\begin{equation}
\frac{\partial U_{\alpha}}{\partial t}(y, t) - \sum_{\beta} \int_{y}^{1} w_{\beta}(z, t) \frac{\partial U_{\beta}}{\partial z}(z, t) dz \frac{\partial U_{\alpha}}{\partial y}(y, t) = \int_{y}^{1} w_{\alpha}(z, t) \frac{\partial U_{\alpha}}{\partial z}(z, t) dz,
\end{equation}
where $\alpha = 1, 2, \ldots$, $(y, t) \in [0, 1] \times [0, T]$.

Note that the equation is now non-local and contains integration. If $w_{\alpha}$ are independent of $y$, then, with (2), (5) reduces to (1).

The equation of the form (1) is known to be solved by considering characteristic curves [2], a curve $y = y_C(t)$ whose derivative is equal to the velocity of fluid;
\begin{equation}
\frac{dy_C}{dt}(t) = \sum_{\beta} w_{\beta}(t) \varphi_\beta(t)
\end{equation}
where $\varphi_{\alpha}(t) = U_{\alpha}(y_C(t), t)$. Then (1) implies an ordinary differential equation for $\varphi_{\alpha}(t)$, which can be solved explicitly, and (6) then implies
\begin{equation}
y_C(t) = 1 - \sum_{\beta} U_{\beta}(y_0, 0) \exp(- \int_{t_0}^{t} w_{\beta}(u) du).
\end{equation}
A natural generalization of (6) for (5) is
\begin{equation}
\frac{dy_C}{dt}(t) = - \sum_{\beta} \int_{y_C(t)}^{1} w_{\beta}(z, t) \frac{\partial U_{\beta}}{\partial z}(z, t) dz.
\end{equation}
As we will see in the present paper, this is no longer solved in such simple form as (7). Introduction of spacial dependence for $w_{\alpha}$ complicates the solution when combined with the boundary condition (2) which conserves component volumes. We will show later that $y_C$ is determined as a fixed point to the map $G$ defined by (71) and (72), a result which apparently deviates largely from (7). To be specific, our proof in § 5 of Theorem 7 proves an expression $y_C = \lim_{n \to \infty} G^n(\theta_0)$ for the characteristic curves, where $\theta_0$ is a constant flow.
In the preceding work [9], the problem (5) for \( w_\alpha \) with spacial dependence was considered under the condition
\[
\sup_{\alpha} \sup_{(y,t)} w_\alpha(y,t) < \infty, \quad \text{and} \quad \sup_{\alpha} \sup_{(y,t)} \frac{\partial w_\alpha}{\partial y}(y,t) < \infty.
\]
In view of (4) for \( \{w_\alpha\} \) with spacially independent case, a natural restriction for \( w_\alpha \) is expected to be a milder one,
\[
\sum_{\alpha} r_\alpha \sup_{(y,t)} w_\alpha(y,t) < \infty, \quad \text{and} \quad \sup_{\alpha} \sup_{(y,t)} \frac{\partial w_\alpha}{\partial y}(y,t) < \infty,
\]
allowing, in particular, fluid mixture with unbounded evaporation rates. The unique existence of the solution was proved for the case of bounded evaporation rates in [9], but explicit formula such as (72) were absent. A main interest in [9] is on the stochastic ranking process [5, 4, 3, 11, 10], and a rather strong restriction on \( w_\alpha \) was posed to prove existence of the hydrodynamic limit of the process with spacially dependent \( w_\alpha \), and (5) appears as the equation which characterizes the limit of the process. In the present paper we focus on (5) itself, and solve the equation under a natural assumption (9). Besides mathematical naturalness of the assumption (9), removal of boundedness condition on \( \{w_\alpha\} \) has practical meaning also on an application of stochastic ranking process to an analysis on a behavior of web ranking data for on-line retail businesses. See [8, 6, 7], in addition to the references above for details on practical applications to web ranking.

As we will show in this paper, the solution \( U_\alpha \) turns out to have a concise expression using the stochastic processes \( N_{\theta,w,z} \) which we introduce in § 3,
\[
U_\alpha(y,t) = \int_{z \in [y_0,1]} \mathbb{P}[N_{y_C,w_\alpha,z}(t) = N_{y_C,w_\alpha,z}(t_0)] \mu_{0,\alpha}(dz),
\]
where \( \mu_{0,\alpha} \) denotes initial spacial distribution of the fluid component \( \alpha \), and \( (y_0,t_0) \) is an initial/boundary point such that the characteristic curve starting from the point satisfies \( y = y_C(t) \). (See (82) with (28).) The map \( G \) of (71) and (72) also has a corresponding expression (70). The processes \( N_{\theta,w,z} \) may be regarded as generalizations of the Poisson process, but, in contrast to the Poisson process, lacks independent increment properties, resulting in the complexity of the solution. In the case of spacially independent evaporation rates, this underlying process reduces to the Poisson process, whose independent increment property implies simple explicit formula such as (7).

We mentioned earlier that the characteristic curve, which is the key quantity for a solution to a one dimensional first order quasilinear partial differential equation, will no longer be obtained by ordinary differential equation for the spacially dependent \( \{w_\alpha\} \), and that it is determined as the fixed point to a map. The map is on the collection of the characteristic curves parametrized by its intersection point with the initial/boundary points, the totality of which we introduce as flow in § 4.

These notions were absent in the preceding work [9], and it is to clarify such mathematical structure of the solution that mainly motivated the present paper.

The plan of the paper is as follows. In § 2 we give the precise statement of our result, where we generalize (5) to allow also for uncountable number of fluid components, by generalizing the unknown functions to measure valued function. In § 3 we introduce the underlying stochastic process and its elementary properties, with which we give an expression of the solution in § 4 (see (82)), assuming existence of a fixed point to a certain map (Theorem 6). The existence of the fixed point is proved in § 5, which completes the existence proof of the solution. A uniqueness proof of the solution is given in § 6. As a remark concerning the condition in (9) on the spacial derivatives of \( w_\alpha \), we apply Schauder’s fixed point theorem in § A to the map defined by (71) and (72), with the condition on derivative relaxed to a global bound on oscillation of \( w_\alpha \).
2 Main Result.

Throughout this paper we fix $T > 0$, $W \subset C^1([0,1] \times [0,T]; [0,\infty))$ a set of non-negative valued $C^1$ functions on $[0,1] \times [0,T]$, and a Borel probability measure $\lambda$ supported on the Borel measurable space $(W, \mathcal{B}(W))$. $\mathcal{B}(W)$ is the $\sigma$-algebra generated by open sets with the topology from the space of continuous functions $C^0([0,1] \times [0,T]; [0,\infty)) \supset C^1([0,1] \times [0,T]; [0,\infty))$ with the metric given by the supremum norm

$$\|w\|_T = \sup_{(y,t) \in [0,1] \times [0,T]} |w(y, t)|.$$  

We assume that

$$M_W := \int_W \|w\|_T \lambda(dw) < \infty$$

and

$$C_W := \sup_{w \in W} \left\| \frac{\partial w}{\partial y} \right\|_T < \infty$$

hold.

Denote the sets of ‘initial ($t = 0$) points’ in the space-time $[0,1] \times [0,T]$, the set of ‘upper stream boundary ($y = 0$) points’, and their union, the set of initial/boundary points, respectively by

$$\Gamma_b = \{0\} \times [0,T] = \{(0,s) \mid 0 \leq s \leq T\},$$

$$\Gamma_i = [0,1] \times \{0\} = \{(z,0) \mid 0 \leq z \leq 1\},$$

$$\Gamma = \Gamma_b \cup \Gamma_i.$$  

(14)

For $t \in [0,T]$, denote the set of initial/boundary points up to time $t$ by

$$\Gamma_t = \{(z, s) \in \Gamma \mid t_0 \leq t\} = \Gamma_i \cup \{(0, t_0) \in \Gamma_b \mid 0 \leq t_0 \leq t\},$$

and the set of admissible pairs of the initial/boundary point $\gamma$ and time $t$ by

$$\Delta_T := \{(\gamma, t) \in \Gamma_T \times [0,T] \mid \gamma \in \Gamma_t\}.$$  

(15)

To state the initial condition, let $\mu_0 = \mu_0(dw \times dz)$ be a Borel probability measure on the measurable space $(W \times [0,1], \mathcal{B}(W \times [0,1]))$ of the product space of $W$ and $[0,1]$. We assume that $\mu_0$ is absolutely continuous with respect to the product measure $\lambda \times dz$, where $dz$ denotes the standard Lebesgue measure on $\mathbb{R}$. Denote the density function by $\sigma$, namely,

$$\int_W \sigma(w,z) \lambda(dw) dz = 1, \quad (w,z) \in W \times [0,1].$$

(17)

We assume $\mu_0(W \times dz) = dz$ and $\mu_0(dw \times [0,1]) = \lambda$, or equivalently, in terms of $\sigma$, we assume

$$\int_W \sigma(w,y) \lambda(dw) = 1, \quad y \in [0,1],$$

and

$$\int_0^1 \sigma(w,z) dz = 1, \quad w \in W.$$  

We now state the main result we prove in this paper. For notational convenience, in the following, and throughout the paper, we use a notation such as $\mu(dw) = \nu(dw)$ to indicate the equality of measures, $\mu(B) = \nu(B)$, for all $B \in \mathcal{B}(W)$.
Theorem 1  There exists a unique pair of functions \( y_C \) and \( \mu_t(dw \times dz) \), where \( y_C \) is a function of \( (\gamma, t) \in \Delta_T \) taking values in \([0, 1]\), and \( \mu_t(dw \times dz) \) is a function of \( t \in [0, T] \) taking values in the probability measures on \( W \times [0, 1] \), such that the following hold.

(i) \( y_C((y_0, 0), t) \) is non-decreasing in \( y_0 \), \( y_C((0, t_0), t) \) is non-increasing in \( t_0 \), and \( y_C(\gamma, t) \) is non-decreasing in \( t \).

(ii) \( y_C(\gamma, t) \) and \( \frac{\partial y_C}{\partial t}(\gamma, t) \) are continuous, and for each \( t \in [0, T] \), \( y_C(\cdot, t) : \Gamma_t \rightarrow [0, 1] \) is surjective.

(iii) For all bounded measurable \( h : W \rightarrow \mathbb{R} \), \( \int_W h(w)\mu_t(dw \times [y, 1)) \) is Lipschitz continuous in \( (y, t) \in [0, 1] \times [0, T] \), with Lipschitz constant uniform in \( h \) satisfying

\[
\sup_{w \in W} |h(w)| \leq 1. 
\]

More precisely,

\[
\left| \int_W h(w)\mu_t(dw \times [y', 1)) - \int_W h(w)\mu_t(dw \times [y, 1)) \right| \leq |y' - y| + M_w e^{2CT}|t' - t|,
\]

for \( h \) satisfying (20).

(iv) The following equation of motion and initial and boundary conditions hold.

\[
y_C((y_0, t_0), 0) = y_0, \quad (y_0, t_0) \in \Gamma, \quad \text{and} \quad \mu_0(dw \times dy) \quad \text{as in} \ (17),
\]

\[
\mu_t(dw \times [0, 1)) = \lambda(dw), \quad t \in [0, T], \tag{23}
\]

\[
\mu_t(W \times [y, 1)) = 1 - y, \quad (y, t) \in [0, 1] \times [0, T], \tag{24}
\]

\[
\mu_t(dw \times [y_C((y_0, t_0), t), 1)) = \mu_{t_0}(dw \times [y_0, 1)) - \int_{t_0}^t \int_{z \in [y_C((y_0, t_0), s), 1]} w(z, s)\mu_s(dw \times dz) ds,
\]

\[
((y_0, t_0), t) \in \Delta_T. \tag{25}
\]

\[\diamondsuit\]

Note that a substitution \( y = y_C(\gamma, t) \) in (24) implies

\[
y_C(\gamma, t) = 1 - \mu_t(W \times [y_C(\gamma, t), 1)), \tag{26}
\]

with which (25) and (24) imply

\[
y_C(\gamma, t) = y_0 + \int_{t_0}^t \int_{W \times [y_C(\gamma, s), 1]} w(z, s)\mu_s(dw \times dz) ds. \tag{27}
\]

If \( W \) is a countable set \( W = \{w_1, w_2, \ldots\} \), denote the distribution functions by

\[
U_\alpha(y, t) = \mu_t(\{w_\alpha\} \times [y, 1)). \tag{28}
\]

Assume further that the functions \( U_\alpha : [0, 1] \times [0, T] \rightarrow [0, \infty) \) are in \( C^1 \). Differentiating (27) by \( t \) we reproduce (8) in § 1. Differentiating (25) by \( t \), substituting (8), and then changing the notation from \( y_C(\gamma, t) \) to \( y \), we can eliminate the dependence on initial/boundary parameter \( \gamma \), and
we reproduce (5) in § 1. With \( r_{\alpha} \) = \( r_{\alpha}, \) (23) and (24) respectively correspond to (2) and (3), and the conditions (12) and (13) imply (9). Thus Theorem 1 contains a solution to the problem introduced in § 1.

In Theorem 1 we claim differentiability for \( yC(\gamma, t) \) in \( t \), while we formulated (25) so that differentiability assumptions on \( U_{\alpha}(y, t) \) or \( \mu_\alpha dw \times [y, 1) \) are absent. In fact, at \( (y, t) \) with \( y = yC((0, 0), t) \), where the characteristic curves starting at initial points \( \gamma \in \Gamma_i \) and those starting at boundary points \( \gamma \in \Gamma_b \) meet, the differentiability with respect to variables which cross the curve are lost in general. Loss of regularity across the characteristic curves is common for the quasilinear partial differential equations [2]. In terms of [2, §3.4], we may therefore say that Theorem 1 claims global existence of the Lipschitz solution (broad solution which is Lipschitz continuous) to the system of quasilinear partial differential equations (5), where we extended the definition of Lipschitz solution in [2, §3.4], to include the non-local (integration) terms, and also generalized the notion of domain of determinancy defined in [2, §3.4], which in the present case corresponds to \( \{(y, t) \in [0, 1] \times [0, \infty) \mid y \geq yC((0, 0), t)\} \), to the boundary condition dependent domain \( \{(y, t) \in [0, 1] \times [0, \infty) \mid y < yC((0, 0), t)\} \). By formulating Theorem 1 in terms of probability measures on \( W \times [0, 1] \) we also included uncountably many components parametrized by the evaporation rates \( w \), which are componentwise bounded but may be unbounded as a total fluid.

3 Point process with last-arrival-time dependent intensity.

Let \( N = N(t), t \geq 0 \), be a non-decreasing, right-continuous, non-negative integer valued stochastic process on a measurable space with \( N(0) = 0 \), and for each non-negative integer \( k \) define its \( k \)-th arrival time \( \tau_k \) by

\[
\tau_k = \inf\{t \geq 0 \mid N(t) \geq k\}, \quad k = 1, 2, \ldots, \quad \text{and} \quad \tau_0 = 0.
\]

The arrival times \( \tau_k \) are non-decreasing in \( k \), because \( N \) is non-decreasing, and since \( N \) is also right-continuous, the arrival times are stopping times; if we denote the associated filtration by \( \mathcal{F}_t = \sigma[N(s), s \leq t] \), then \( \{\tau_k \leq t\} \in \mathcal{F}_t, t \geq 0 \).

Let \( \omega \) be a non-negative valued bounded continuous function of \( (s, t) \) for \( 0 \leq s \leq t \), and for \( k = 1, 2, \ldots \) assume that

\[
P[t < \tau_k \mid \mathcal{F}_{\tau_{k-1}}] = \exp(-\int_{\tau_{k-1}}^{t} \omega(\tau_{k-1}, u) \, du) \quad \text{on} \quad t \geq \tau_{k-1}.
\]

In particular, (30) with \( k = 1 \) implies

\[
P[N(t) = 0] = P[\tau_1 > t] = \exp(-\int_0^t \omega(0, u) \, du), \quad t \geq 0.
\]

Note that the function \( \omega \) has different dependence on the variables from the evaporation rate function \( w \) in the other sections of this paper. (We will relate \( \omega \) to \( w \) by (51) in § 4, namely, we will introduce an intensity function as a composite function of the evaporation rate function and a flow.) If \( \omega \) is independent of the first variable, then (30) implies that \( N \) is the (inhomogeneous) Poisson process with intensity function \( \omega \). We are considering a generalization of the Poisson process such that the intensity function depends on the latest arrival time.

The remainder of this section is devoted to basic formulas to be used in this paper. For a continuously differentiable function \( f \) vanishing at \( \infty \), (30), with integration by parts and the
Fubini’s theorem implies

\[
\int_{\tau_{k-1}}^{\infty} f(t) \omega(\tau_{k-1}, t) \exp(- \int_{\tau_{k-1}}^{t} \omega(\tau_{k-1}, s) \, ds) \, dt
\]

\[
= \int_{\tau_{k-1}}^{\infty} f'(t) \exp(- \int_{\tau_{k-1}}^{t} \omega(\tau_{k-1}, s) \, ds) \, dt + f(\tau_{k-1})
\]

(32)

where we put

\[
\tau
\]

By induction and (32) implies

\[
E[ \int_{\tau_{k-1}}^{\infty} f'(t) \, dt \mid \mathcal{F}_{\tau_{k-1}} ] + f(\tau_{k-1})
\]

(33)

and define a linear map \( A_\omega : L_0([0, \infty)) \to L_0([0, \infty)) \) by

\[
(A_\omega f)(t) = \int_{t}^{\infty} f(u) \omega(t, u) e^{-\Omega(t,u)} \, du.
\]

Then (32) implies

\[
E[ f(\tau_k) \mid \mathcal{F}_{\tau_{k-1}} ] = (A_\omega f)(\tau_{k-1}), \quad f \in L_0([0, \infty)).
\]

By induction and \( \tau_0 = 0 \) we have

\[
E[ f(\tau_k) ] = E[ E[ \cdots E[ E[ f(\tau_k) \mid \mathcal{F}_{\tau_{k-1}} ] \mid \mathcal{F}_{\tau_{k-2}} ] \cdots \mid \mathcal{F}_{\tau_1} ] ]
\]

(34)

\[
= (A_k \omega f)(0)
\]

(35)

\[
= \int_{0 \leq u_1 \leq u_2 \leq \cdots \leq u_k < \infty} f(u_k) \prod_{i=1}^{k} \omega(u_{i-1}, u_i) e^{-\Omega(u_{i-1},u_i)} \, du,
\]

where we put \( u_0 = 0 \) to simplify notations.

For example, by choosing \( f(u) = 1_{u \leq t} \), (35) implies

\[
P[ \tau_k \leq t \mid \mathcal{F}_{\tau_{k-1}} ] = (1 - e^{-\Omega(\tau_{k-1},t)}) 1_{\tau_{k-1} \leq t}.
\]

Then (37) and (36) with \( f(u) = e^{-\Omega(u,t)} 1_{u \leq t} \) imply

\[
P[ N(t) = k ] = P[ \tau_k \leq t < \tau_{k+1} ] = E[ 1_{\tau_k \leq t} (1 - P[ \tau_{k+1} \leq t \mid \mathcal{F}_{\tau_k} ]) ]
\]

(38)

\[
= E[ f(\tau_k) ] = (A_k \omega f)(0), \quad k \in \mathbb{Z}_+, \ t > 0.
\]

Hence, as in the last line of (36),

\[
P[ N(t) = k ] = (A_k \omega f)(0)
\]

(39)

\[
= \left\{ \begin{array}{ll}
  \int_{0 \leq u_1 \leq u_2 \leq \cdots \leq u_k \leq t} e^{-\Omega(u_{k-1},u_k)} \prod_{i=1}^{k} \omega(u_{i-1}, u_i) e^{-\Omega(u_{i-1},u_i)} \, du, & k = 1, 2, \ldots, \\
  e^{-\Omega(0,t)}, & k = 0.
\end{array} \right.
\]
In particular, \( P[ N(t) \geq 0 ] = 1 \) implies a sum rule

\[
e^{-\Omega(0,t)} + \sum_{k=1}^{\infty} \int_{0 \leq u_1 \leq u_2 \leq \ldots \leq u_k \leq t} e^{-\Omega(u_k,t)} \prod_{i=1}^{k} \omega(u_{i-1}, u_i) e^{-\Omega(u_{i-1}, u_i)} du_i = 1,
\]

where \( u_0 = 0 \), as in (36).

Similarly, given \( s \) and \( t \) satisfying \( 0 \leq s < t \), the probability that there is no arrival in the interval \((s, t] \) is

\[
P[ N(t) = N(s) ] = \sum_{k=0}^{\infty} P[ N(t) = N(s) = k ] = \sum_{k=0}^{\infty} P[ \tau_k \leq s, t < \tau_{k+1} ].
\]

(41)

With \( f(u) = 1_{u \leq s} e^{-\Omega(u,t)} \) in (36), we also have an explicit formula

\[
P[ N(t) = N(s) = k ] = \mathbb{E}[ 1_{\tau_k \leq s} e^{-\Omega(\tau_k,t)} ] = (A^k_u f)(0)
\]

(42)

for \( t \geq s > 0 \), where \( u_0 = 0 \), as in (36). The following property relates the \( s \) and \( t \) dependencies of the quantity in (41). Note that the explicit formula (41) implies that this quantity is \( C^1 \) in \( s \) and \( t \).

**Proposition 2** For \( k = 1, 2, \ldots \),

\[
\frac{\partial}{\partial t} P[ N(t) = N(s) = k ] = - \int_{0}^{s} \omega(u, t) \frac{\partial}{\partial u} P[ N(t) = N(u) = k ] du,
\]

\( 0 \leq s < t. \)

\( \Diamond \)

**Proof.** First we prove

\[
\mathbb{E}[ f(\tau_k) g(\tau_k) 1_{\tau_k \leq s} ] = \int_{0}^{s} f(u) Q'(u) du
\]

(44)

for locally bounded and measurable \( f \) and \( g \) such that

\[
Q(s) := \mathbb{E}[ g(\tau_k) 1_{\tau_k \leq s} ]
\]

is absolutely continuous with respect to the Lebesgue measure (so that the derivative \( Q' \) almost surely exists). Approximating by a series of smooth functions, it suffices to prove (44) for \( f \in C^1 \). By Fubini’s theorem and partial integration, and noting that \( \tau_k > 0 \) for \( k > 0 \) implies \( Q(0) = 0 \),

\[
\mathbb{E}[ f(\tau_k) g(\tau_k) 1_{\tau_k \leq s} ] = \mathbb{E}[ \left( - \int_{\tau_k}^{s} f'(u) du + f(s) \right) g(\tau_k) 1_{\tau_k \leq s} ]
\]

\[
= f(s) Q(s) - \int_{0}^{s} f'(u) \mathbb{E}[ 1_{\tau_k \leq u} g(\tau_k) ] du
\]

\[
= f(s) Q(s) - \int_{0}^{s} f'(u) Q(u) du
\]

\[
= \int_{0}^{s} f(u) Q'(u) du.
\]
Thus (44) is proved.

Now for a positive integer \( k \), let \( f(u) = \omega(u, t) \) and \( g(u) = e^{-\Omega(u, t)} \) in (44). Note that for this choice (42) implies

\[
Q(s) = \mathbb{E}[ e^{-\Omega(\tau_k, t)} \mathbf{1}_{\tau_k \leq s} ] = \mathbb{P}[ N(t) = N(s) = k ].
\]

Then (44) implies

\[
\frac{\partial}{\partial t} \mathbb{P}[ N(t) = N(s) = k ] = -\mathbb{E}[ \omega(\tau_k, t) e^{-\Omega(\tau_k, t)} \mathbf{1}_{\tau_k \leq s} ]
\]

\[
= -\mathbb{E}[ f(\tau_k) g(\tau_k) \mathbf{1}_{\tau_k \leq s} ] = -\int_0^s \mathbb{E}[ N(t) = N(u) = k ] du,
\]

which proves (43).

Note that, for example, the explicit formula in (42) depends on the intensity function \( \omega \) at times before \( s \), which implies that the process \( N \) is not of independent increment, hence, in particular, is not a Poisson process. If, on the other hand, \( \omega \) is independent of its first variable, put \( \tilde{\omega}(t) = \omega(s, t) \).

Then (33) implies \( \Omega(t_0, t) = \int_{t_0}^t \tilde{\omega}(s) ds \), and (42) is simplified as

\[
\mathbb{P}[ N(t) = N(s) ] = \left( 1 + \sum_{k \geq 1} \int_{0 \leq u_1 \leq u_2 \leq \cdots \leq u_k \leq s} \prod_{i=1}^k \mathbb{E}[ N(t) = N(u) = k ] du \right) e^{-\Omega(0, t)}
\]

\[
= \left( 1 + \sum_{k \geq 1} \frac{1}{k!} \Omega(0, s)^k \right) e^{-\Omega(0, t)}
\]

\[
= e^{\Omega(0, s)} \times e^{-\Omega(0, t)} = e^{-\Omega(s, t)},
\]

where we used an elementary formula proved by induction in \( k \),

\[
\int_{0 \leq u_1 \leq u_2 \leq \cdots \leq u_k \leq s} \prod_{i=1}^k f(u_i) du_1 du_2 \ldots du_k = \frac{1}{k!} \left( \int_0^s f(v) dv \right)^k, \quad s \geq 0, \quad k = 1, 2, \ldots,
\]

valid for any integrable function \( f : \mathbb{R} \to \mathbb{R} \). The simple result (46) reproduces a formula for the (inhomogeneous) Poisson process with independent increments. In the general case of processes we consider in this paper, such simple relations to Poisson processes or Poisson distributions are lost and the properties of the processes become complicated.

### 4 Flows and construction of solution.

The key quantities for the solution to the functional equations in Theorem 1 are the characteristic curves \( y_C \) and the associated measure \( \varphi(dw, \gamma, t) = \mu_t(dw \times [y_C(\gamma, t), 1]) \). We will find \( y_C \) as a unique solution to a non-linear map on a space \( \Theta_T \) of flows, a non-decreasing function in time \( t \) and in initial/boundary points \( \gamma \in \Gamma \). To simplify the definition of \( \Theta_T \) we first define a total order \( \succeq \) on \( \Gamma \) by

\[
s \leq t, \quad z \leq y \iff (0, T) \succeq (0, t) \succeq (0, s) \succeq (0, 0) \succeq (z, 0) \succeq (y, 0) \succeq (1, 0).
\]
We now define the set of flows $\Theta_T$ on $[0,1] \times [0,T]$ by

$$\Theta_T := \{ \theta : \Delta_T \to [0,1] \mid \theta((y_0,t_0),t_0) = y_0, (y_0,t_0) \in \Gamma_T, \text{ continuous, }$$

surjective and non-increasing in $\gamma$ for each $t$,
non-decreasing in $t$ for each $\gamma \}.$$  

(49) For example,

$$\theta((1,0),t) = 1, \quad t \in [0,T], \quad \theta \in \Theta_T.$$  

(50) Let $W$, the set of evaporation rates, be as in Theorem 1, and let $\theta \in \Theta_T$. For each $w \in W$ and $z \in [0,1)$ define $\omega = \omega_{\theta,w,z}$, a non-negative valued continuous function of $(s,t)$ satisfying

$$0 \leq s \leq t \leq T, \quad \text{by}$$

$$\omega_{\theta,w,z}(s,t) = \begin{cases} w(\theta((z,0),t),t), & \text{if } s = 0, \\ w(\theta((0,s),t),t), & \text{if } s > 0. \end{cases}$$  

(51) Note that $\omega_{\theta,w,z}$ is independent of $z$ if $s > 0$. Let $\{N_{\theta,w,z} \mid z \in [0,1), w \in W\}$ be a set of processes, with each $N_{\theta,w,z}$ being a point process $N$ introduced in § 3 with the intensity function in (30) determined by $\omega = \omega_{\theta,w,z}$. The quantity in (33) for the choice (51) is

$$\Omega_{\theta,w,z}(0,t) = \int_0^t w(\theta((z,0),u),u) du, \quad 0 \leq s \leq t \leq T,$$

$$\Omega_{\theta,w}(s,t) = \int_s^t w(\theta((0,s),u),u) du, \quad 0 < s \leq t.$$  

(52) Let $\mu_0$ be as in Theorem 1, and define a function $\varphi_0(dw,\gamma,t)$ on $(\gamma,t) \in \Delta_T$ taking values in the measures on $W$, by

$$\varphi_0(dw,\gamma,t) = \int_{z \in [y_0,1)} P[N_{\theta,w,z}(t) = \nu_{\theta,w,z}(t_0) \mid \mu_0(dw \times dz), \gamma = (y_0,t_0) \in \Gamma, (\gamma,t) \in \Delta_T].$$  

(53) The explicit form for (53) is simple for $\gamma = (y_0,0) \in \Gamma_1$, because $N_{\theta,w,z}(0) = 0$, and (42) with $k = 0$ imply

$$\varphi_0(dw,(y_0,0),t) = \int_{z \in [y_0,1)} e^{-\Omega_{\theta,w,z}(0,t)} \mu_0(dw \times dz).$$  

(54) For $\varphi_0$ in (53) define $\frac{\partial \varphi_0}{\partial \gamma}$, a measure valued function on $\Delta_T$, by

$$\frac{\partial \varphi_0}{\partial \gamma}(dw,\gamma,t) = \begin{cases} \frac{\partial \varphi_0}{\partial z}(dw,(z,0),t), & \text{if } \gamma = (z,0) \in \Gamma_i, \\ \frac{\partial \varphi_0}{\partial u}(dw,\gamma,t), & \text{if } \gamma = (0,u) \in \Gamma_b. \end{cases}$$  

(55) We keep non-negativity of the defined measure in determining the sign. Explicit calculation of the derivative at $\gamma \in \Gamma_i$ is straightforward from (54) and (17). The derivative at $\gamma = (0,u) \in \Gamma_b$ is also
calculated explicitly using (42) and (53), which is

\[
\frac{\partial \varphi}{\partial u}(dw, (0, u), t) = \int_{z \in [0,1]} w(\theta((z, 0), u) e^{-\Omega_{\theta,w,z}(0,u)} + \sum_{k \geq 2} \int_{0 < u_1 \leq u_2 \leq \cdots \leq u_{k-1} \leq u} w(\theta((z, 0), u_1) e^{-\Omega_{\theta,w,z}(0,u_1)} du_1 \\
\prod_{i=2}^{k-1} (w(\theta((0, u_{i-1}), u_i) e^{-\Omega_{\theta,w,z}(u_{i-1},u_i)} du_i) \\
w(\theta((0, u_{k-1}), u) e^{-\Omega_{\theta,w,z}(u_{k-1},u)}) e^{-\Omega_{\theta,w,z}(u,t)} \mu_0(dw \times dz),
\]

(56)

where we also used the notations (51) and (52) to make the \( z \) and \( u \) dependence explicit. Note that \( w \in W \) are non-negative functions, hence \( e^{-\Omega_{\theta,w,z}(u,t)} \leq 1 \). This and the sum rule (40) (with the replacements \( t = u \) and \( k = k' - 1 \), with (12), (17), and (19) imply

\[
\frac{\partial \varphi}{\partial u}(W; (0, u), t) \leq \int_{B \times [0,1]} \|w\|_T \mu_0(dw \times dz) = \int_B \|w\|_T \lambda(dw) = M_W < \infty,
\]

hence, \( \frac{\partial \varphi}{\partial u} \) is well-defined.

For \( f : \Gamma_T \to \mathbb{R} \) and a Borel subset \( A \subset \Gamma_T \), define \( \int_A f(\gamma) d\gamma \), a line integral on \( \Gamma_T \), by

\[
\int_A f(\gamma) d\gamma = \int_{A_i} f(z, 0) dz + \int_{A_b} f(0, u) du,
\]

(57)

where, \( A_i = \{ z \in [0,1] \mid (z, 0) \in A \} \) and \( A_b = \{ u \in [0,T] \mid (0, u) \in A \} \), and the integration in the right hand side of (57) are the standard one dimensional integrations.

**Proposition 3** It holds that

\[
0 \leq \int_{\gamma' \in \Gamma_t} \int_W \|w\|_T \frac{\partial \varphi}{\partial u}(dw, \gamma', t) d\gamma' \leq M_W e^{2C_W t},
\]

(58)

for all \( t \in [0,T] \).

\( \diamond \)

**Remark.** Note the extra \( \|w\|_T \) in the integrand of (58). We are allowing \( W \) to contain unbounded functions \( w \), so the finiteness of (58) is harder than that of (56). We use the condition (13) as well as (12) to prove (58).

Proposition 3 implies, in particular, that the integration (63) introduced later, is uniformly bounded in \( (\gamma, t) \in \Delta_T \) and \( B \in B(W) \).

\( \diamond \)

**Proof of Proposition 3.** By the definitions (55) and (57),

\[
\int_{\gamma' \in \Gamma_t} \int_W \|w\|_T \frac{\partial \varphi}{\partial u}(dw, \gamma', t) d\gamma' = -\int_0^1 \int_W \|w\|_T \frac{\partial \varphi}{\partial z}(dw, (z, 0), t) dz + \int_0^1 \int_W \|w\|_T \frac{\partial \varphi}{\partial u}(dw, (0, u), t) du.
\]

(59)
The first term on the right hand side is explicitly calculated using (54) and (17). Using also $e^{-\Omega_{\theta,w,z}(0,t)} \leq 1$, (19), and (12), we have an estimate

\begin{equation}
0 \leq -\int_0^1 \int_W \|w\|_T \frac{\partial \varphi_{\theta}}{\partial z}(dw,(z,0),t)\,dz = \int_W \|w\|_T e^{-\Omega_{\theta,w,z}(0,t)} \mu_0(dw \times dz)
\end{equation}

\begin{equation}
\leq MW < \infty.
\end{equation}

Next, for $t \geq s \geq 0$ and $w \in W$, put

\begin{equation}
\tilde{\Omega}_w(s,t) = \int_s^t w(1,u)\,du.
\end{equation}

Then the condition (13) and the fact that $\theta$ takes values in $[0,1]$ imply

\begin{equation}
|w(\theta(\gamma,t),t) - w(1,t)| \leq CW,
\end{equation}

\begin{equation}
|\Omega_{\theta,w}(s,t) - \Omega_{\theta,w}(s,t)| \leq CW(t-s), \quad \text{and} \quad |\Omega_{\theta,w,z}(0,t) - \tilde{\Omega}_w(0,t)| \leq CW t,
\end{equation}

$(\gamma,t) \in \Delta_T, 0 \leq s \leq t, w \in W$.

Using (62) and (61) in (56), and then using (17) and (19), we have an estimate

\begin{equation}
\int_0^t \int_W \|w\|_T \frac{\partial \varphi_{\theta}}{\partial u}(dw,(0,u),t)\,du \\
\leq \int_W \|w\|_T e^{-\tilde{\Omega}_w(0,t) + CW t} \int_0^t (w(1,u) + CW) \left(1 + \sum_{k \geq 1} \int_{0 \leq u_1 \leq \ldots \leq u_k \leq u} \prod_{i=1}^k ((w(1,u_i) + CW)\,du_i)\right)\,du \lambda(dw).
\end{equation}

The estimate is now reduced to that for the Poisson processes, and (47) implies

\begin{equation}
\int_0^t \int_W \|w\|_T \frac{\partial \varphi_{\theta}}{\partial u}(dw,(0,u),t)\,du \\
\leq \int_W \|w\|_T e^{-\tilde{\Omega}_w(0,t) + CW t} \int_0^t e^{\tilde{\Omega}_w(0,u) + CW u}(w(1,u) + CW u)\,du \lambda(dw) \\
= \int_W \|w\|_T e^{-\tilde{\Omega}_w(0,t) + CW t} e^{\tilde{\Omega}_w(0,u) + CW u}\bigg|_{u=0}^{u=t}\lambda(dw) \\
\leq e^{2CW t} \int_W \|w\|_T \lambda(dw) \leq MW e^{2CW t}.
\end{equation}

This proves (58). \qed

**Proposition 4** It holds that

\begin{equation}
\frac{\partial \varphi_{\theta}}{\partial t}(B,\gamma,t) = -\int_{\gamma \geq \gamma'} \int_B w(\theta'(\gamma',t),t) \frac{\partial \varphi_{\theta}}{\partial \gamma}(dw,\gamma',t)\,d\gamma',
\end{equation}

$(\gamma,t) \in \Delta_T, B \in \mathcal{B}(W),$ where $\gamma \geq \gamma'$ is defined in (48). \qed
**Remark.** If $W$ consists of functions with no spatial dependence, namely, if $w(y,t) = w(1,t)$, then the factor $w$ in the integrand of the right hand side of (63) is constant for $\gamma'$ integration, and we have integration after differentiation, so that the right hand side is simplified as $-w(1,t)\varphi_\theta(B,\gamma,t)$, and the equation is solved easily, as remarked below (6) in § 1. 

**Proof of Proposition 4.** Consider first the case $\gamma = (y_0,0) \in \Gamma_i$. The explicit form (54), together with the definitions (55), (57), and (17), implies

$$
- \int_{\gamma \geq \gamma'} \int_B w(\theta(\gamma',t),t) \frac{\partial \varphi_\theta}{\partial \gamma}(dw,\gamma',t) \, d\gamma' \\
= - \int_{z \in [y_0,1)} \int_B w(\theta((z,0),t),t) e^{-\Omega_{s,w,z}(0,t)} \sigma(w,z) \lambda(dw) \, dz \\
= - \int_{z \in [y_0,1)} \int_B w(\theta((z,0),t),t) e^{-\Omega_{s,w,z}(0,t)} \mu_0(dw \times dz) = \frac{\partial \varphi_\theta}{\partial t}(B,(y_0,0),t),
$$

which proves (63) for $\gamma \in \Gamma_i$.

To prove (63) for $\gamma \in \Gamma_b$, put, for $k \in \mathbb{Z}_+$,

$$
\varphi_\theta^{(k)}(dw,(y_0,t_0),t) = \int_{z \in [y_0,1)} P[N_{s,w,z}(t) = N_{s,w,z}(t_0) = k] \mu_0(dw \times dz).
$$

Then (53) implies

$$
\varphi_\theta = \sum_{k=0}^{\infty} \varphi_\theta^{(k)}.
$$

We will prove, for $\gamma = (0,t_0) \in \Gamma_b$

$$
\frac{\partial \varphi_\theta^{(k)}}{\partial t}(B,\gamma,t) = - \int_{\gamma \geq \gamma'} \int_B w(\theta(\gamma',t),t) \frac{\partial \varphi_\theta^{(k)}}{\partial \gamma}(dw,\gamma',t) \, d\gamma',
$$

for all $k \in \mathbb{Z}_+$, where differentiation and integration with respect to $\gamma$ are defined in accordance with (55) and (57). Then (65) and (66) prove (63). The changes in the order of series and integration and differentiation causes no problem, because all the terms and integrands are non-negative and the results of summation and integration are bounded by Proposition 3.

Consider first the case $k = 0$. Then (42) implies

$$
\varphi_\theta^{(0)}(dw,(y_0,t_0),t) = \int_{z \in [y_0,1)} e^{-\Omega_{s,w,z}(0,t)} \mu_0(dw \times dz).
$$

Note that this is independent of $t_0$. Hence (55) and (17) imply

$$
\frac{\partial \varphi_\theta^{(0)}}{\partial \gamma}(dw,\gamma',t) = \begin{cases} e^{-\Omega_{s,w,z}(0,t)} \sigma(w,z) \lambda(dw), & \text{if } \gamma' = (z,0) \in \Gamma_i, \\
0, & \text{if } \gamma' = (0,u) \in \Gamma_b.
\end{cases}
$$

For the case $\gamma = (0,t_0) \in \Gamma_b$,

$$
\gamma \geq \gamma' \iff \gamma' = (0,u), \ 0 \leq u \leq t_0, \text{ or } \gamma' = (z,0), \ 0 \leq z \leq 1.
$$
The contribution, however, to (57) from the integration along $\Gamma_b$ vanishes because the integrand (68) is 0 on $\Gamma_b$. Hence, (68), (57), (17), and (67) imply
\[
\int_{\gamma'} w(\theta(\gamma', t), t) \frac{\partial \varphi_0(0)}{\partial \gamma'} (dw, \gamma', t) \, d\gamma' \\
= \int_{\gamma \in [0,1]} \int_{B} w(\theta((z, 0), t), t) e^{-\Omega_{\theta,w,z}(0,t)} \sigma(w, z) \lambda(dw) \, dz \\
= \int_{B \times [0,1]} w(\theta((z, 0), t), t) e^{-\Omega_{\theta,w,z}(0,t)} \mu_0(dw \times dz) \\
= - \frac{\partial \varphi_0(0)}{\partial t}(B, \gamma, t),
\]
which proves (66) for $k = 0$ and $\gamma \in \Gamma_b$.

To consider the case $k > 0$ and $\gamma = (0, t_0) \in \Gamma_b$, (64), Proposition 2, and (51) imply
\[
\varphi_k((dw, \gamma, t) = 0, \quad \gamma \in \Gamma_i, \quad t \geq 0, \quad k = 1, 2, \ldots,
\]
hence (66) is proved for this case. This completes a proof of (66), and (63) follows.

Next put
\[
G(\theta)(\gamma, t) = 1 - \varphi_0(W, \gamma, t) \\
= 1 - \int_{W \times [0,1]} P[N_{\theta,w,z}(t) = N_{\theta,w,z}(t_0)] \, \mu_0(dw \times dz) \\
= y_0 + \int_{W \times [0,1]} P[N_{\theta,w,z}(t) > N_{\theta,w,z}(t_0)] \, \mu_0(dw \times dz) \\
\gamma = (y_0, t_0) \in \Gamma, \quad (\gamma, t) \in \Delta_T.
\]
With (17) and (42), we have an explicit formula
\[
G(\theta)((y_0, 0), t) = 1 - \int_{W \times [0,1]} e^{-\Omega_{\theta,w,z}(0,t)} \sigma(w, z) \lambda(dw) \, dz,
\]
\gamma = (y_0, 0) \in \Gamma_i,
and
\[
G(\theta)((0, t_0), t) = 1 - \int_{W \times [0,1]} e^{-\Omega_{\theta,w,z}(0,t)} \sigma(w, z) \lambda(dw) \, dz \\
- \int_{W \times [0,1]} \sum_{k \geq 1} \int_{0 \leq u_1 \leq \cdots \leq u_k \leq t_0} w(\theta((z, 0), u_1), u_1) e^{-\Omega_{\theta,w,z}(0,u_1)} \\
\times \prod_{i=2}^k \left( w(\theta((0, u_{i-1}), u_i), u_i) e^{-\Omega_{\theta,w}(u_{i-1}, u_i)} \right) \\
\times e^{-\Omega_{\theta,w}(u_k,t)} \prod_{i=1}^k du_i \sigma(w, z) \lambda(dw) \, dz, \\
\gamma = (0, t_0) \in \Gamma_b \cap \Gamma_f.
\]
Using these explicit formula with (18) and (40), we see from (49) that \(G(\theta) \in \Theta_T\). Namely, (70) defines a map
\[
G : \Theta_T \to \Theta_T
\]
on the set of flows \(\Theta_T\).

**Proposition 5** If \(\theta \in \Theta_T\) is a fixed point of \(G\) in (73), namely, if \(G(\theta) = \theta\), then \(\varphi_\theta\) defined by (53) uniquely determines for each \(t \in [0, T]\) a probability measure \(\mu_{\theta,t}\) on \(W \times [0, 1]\) by the equation
\[
\varphi_\theta(dw, \gamma, t) = \mu_{\theta,t}(dw \times [\theta(\gamma, t), 1]), \quad (\gamma, t) \in \Delta_T.
\]

Furthermore, we have the following formula for a change of integration variables:
\[
\int_{W \times \{\gamma' \in \Gamma_t | \gamma' \geq \gamma\}} f(w, \theta(\gamma', t), t) \frac{\partial \varphi_\theta}{\partial \gamma}(dw, \gamma', t) d\gamma' = \int_{W \times [\theta(\gamma, t), 1]} f(w, z, t) \mu_{\theta,t}(dw \times dz),
\]
for integrable function \(f : W \times [0, 1] \times [0, T] \to \mathbb{R}\).

**Remark.** We are working with a generalization of characteristic curves, for which \((\gamma, t)\) is a good coordinate. On the other hand, equation of motions are usually stated in the space time coordinates \((y, t)\). This proposition relates the representations in these distinct coordinate systems.

**Proof of Proposition 5.** The definition (49) of \(\Theta_T\) implies that \(\theta(\cdot, t) : \Gamma_t \to [0, 1]\) is continuous, surjective and non-decreasing. Hence to prove (74) we only need to prove consistency, namely,
\[
\theta(\gamma, t) = \theta(\gamma', t) \Rightarrow \varphi_\theta(B, \gamma, t) = \varphi_\theta(B, \gamma', t), \quad B \in \mathcal{B}(W).
\]
If (76) holds, then (74) determines the distribution function \(\mu_{\theta,t}(B \times [y, 1]), y \in [0, 1],\) on \(W \times [0, 1]\), so that \(\mu_{\theta,t}(B \times [a, b])\) is determined, and eventually \(\mu_{\theta,t}\) is determined as a probability measure on the product space \(W \times [0, 1]\).

Assume that \(\theta(\gamma, t) = \theta(\gamma', t)\) for \(\gamma = (y_0, t_0) \in \Gamma_t\) and \(\gamma' = (y_0', t_0') \in \Gamma_t\). Since \((\Gamma, \succeq)\) is a totally ordered set, we may assume without loss of generality that \(\gamma \succeq \gamma'\). Then (48) implies \(t_0 \leq t_0' \leq t\) and \(y_0 \geq y_0'\). Non-decreasing property of the point process \(N_{\theta,w,z}(t)\) in \(t\) and monotonicity of measures imply with the definition (53),
\[
\varphi_\theta(B, \gamma', t) \geq \varphi_\theta(B, \gamma, t), \quad \text{and} \quad \varphi_\theta(B^c, \gamma', t) \geq \varphi_\theta(B^c, \gamma, t),
\]
for any \(B \in \mathcal{B}(W)\).

On the other hand, the assumption \(G(\theta) = \theta\) and (70) and \(\theta(\gamma, t) = \theta(\gamma', t)\) imply
\[
\varphi_\theta(B, \gamma, t) + \varphi_\theta(B^c, \gamma, t) = \varphi_\theta(W, \gamma, t) = 1 - G(\theta)(\gamma, t) = 1 - \theta(\gamma, t)
\]
\[
= 1 - \theta(\gamma', t) = \varphi_\theta(B, \gamma', t) + \varphi_\theta(B^c, \gamma', t).
\]
Hence
\[
\varphi_\theta(B, \gamma', t) - \varphi_\theta(B, \gamma, t) = -(\varphi_\theta(B^c, \gamma', t) - \varphi_\theta(B^c, \gamma, t)).
\]
Combining (77) and (78), we see that (76) holds, which implies (74).
Next to prove (75), we first prove
\[
\int_{\gamma \geq \gamma'} \frac{\partial \varphi_{\theta}}{\partial \gamma}(B, \gamma', t) \, d\gamma' = \mu_{\theta,t}(B \times \theta(\gamma,t), (\gamma,t) \in \Delta_T, B \in \mathcal{B}(W).
\]

In fact, note that (53) implies
\[
\varphi_{\theta}(B, (1,0), t) = 0, \quad B \in \mathcal{B}(W), \quad t \geq 0.
\]

Then, if \(\gamma = (y_0, 0) \in \Gamma_i\), The explicit formula (42), with the definitions (53) (55) (57), implies
\[
\int_{\gamma \geq \gamma'} \frac{\partial \varphi_{\theta}}{\partial \gamma}(B, \gamma', t) \, d\gamma' = -\int_{y_0}^{1} \frac{\partial \varphi_{\theta}}{\partial z}(B, (z,0), t) \, dz
\]
\[
= \varphi(B, (y_0, 0), t),
\]
where we used (80) in the last line. If \(\gamma = (0, t_0) \in \Gamma_b\), the explicit formula (42) similarly implies
\[
\int_{\gamma \geq \gamma'} \frac{\partial \varphi_{\theta}}{\partial \gamma}(B, \gamma', t) \, d\gamma' = -\int_{0}^{t_0} \frac{\partial \varphi_{\theta}}{\partial u}(B, (0,u), t) \, du + \int_{0}^{t_0} \frac{\partial \varphi_{\theta}}{\partial z}(B, (z,0), t) \, dz
\]
\[
= \varphi(B, (0,t_0), t),
\]
hence (79) follows from (74).

Define a measure \(\nu_{\theta,t}\) on \(W \times \Gamma_t\) by
\[
\nu_{\theta,t}(dw \times d\gamma) = \frac{\partial \varphi_{\theta}}{\partial \gamma}(dw, \gamma, t) \, d\gamma.
\]

Then (79) implies
\[
\nu_{\theta,t}(B \times \{\gamma' \mid \gamma \geq \gamma'\}) = \mu_{\theta,t}(B \times \theta(\gamma,t), 1), \quad \gamma \in \Gamma_t.
\]

This implies that, if we put \(X_{\theta,t} = (id_W, \theta)\), where \(id_W\) is the identity map on \(W\), then \(\mu_{\theta,t}\) is the image measure of \(\nu_{\theta,t}\) with respect to the map \(X_{\theta,t} : W \times \Gamma_t \to W \times [0,1]:\)
\[
\mu_{\theta,t} = \nu_{\theta,t} \circ X_{\theta,t}^{-1}.
\]

Therefore (75) follows.

\[\Box\]

**Theorem 6** Assume that \(G\) in (73) has a fixed point, and denote the fixed point by \(y_C \in \Theta_T;\)
\[
y_C = G(y_C)
\]

Put \(\varphi = \varphi_{y_C}\) and \(\mu_t = \mu_{y_C,t}\), where \(\varphi_{\theta}\) and \(\mu_{\theta,t}\) are defined by (53) and (74) with \(\theta = y_C\). Then the so defined \(y_C\) and \(\mu_t\) satisfy all the properties stated in Theorem 1. \(\Diamond\)
Remark. With (53) and (74), the theorem implies an expression

$$
\mu_{g_\gamma,t}(dw \times [y_C(\gamma,t),1]) = \int_{z \in [y_0,1]} P[N_{g_\gamma,w,z}(t) = N_{g_\gamma,w,z}(t_0)] \mu_0(dw \times dz),
$$

$$
\gamma = (y_0,t_0) \in \Gamma_t,
$$

for the solution to Theorem 1.

The properties of the solution claimed in Theorem 1 are mostly contained in the previous propositions and explicit formulas. The remaining point is the Lipschitz continuity of $\mu_t$. If $w(y,t) = 0$ for certain time interval for all $w \in W$, then the characteristic curve $y_C$ will remain constant for the interval, and a small change in $y = y_C(\gamma,t)$ may result in a large change in $\gamma$ and hence in $\mu_t$.

It turns out that the situation causes no problem, because then the change in the quantity $\varphi$ and eventually $\mu_t$ are small in the time interval, hence continuity follows.

$$\Diamond$$

Proof of Theorem 6. By definition (70), we have

$$
y_C(\gamma,t) = G(y_C)(\gamma,t) = 1 - \varphi_{g_\gamma}(W,\gamma,t),
$$

for which monotonicity properties stated in Theorem 1 are direct consequences of (53). Proposition 3 and Proposition 4 with explicit formula (71) and (72), or (53) and (56), imply that $y_C$ and $\frac{\partial y_C}{\partial t}$ is continuous in $(\gamma,t)$.

Let $h : W \rightarrow \mathbb{R}$ be a measurable function satisfying (20). For $(y,t)$, $(y',t') \in [0,1] \times [0,T]$, choose $\gamma \in \Gamma_t$ and $\gamma' \in \Gamma_t$ such that $y = y_C(\gamma,t)$ and $y' = y_C(\gamma',t)$. We may assume $\gamma \geq \gamma'$. Then monotonicity of $y_C$ implies $y \leq y'$, hence $h(w) \leq 1$ and (24) imply

$$
\int_W h(w)\mu_t(dw \times [y,1]) - \int_W h(w)\mu_t(dw \times [y',1]) = \int_W h(w)\mu_t(dw \times [y,y'])
$$

$$
\leq \mu_t(W \times [y,y']) = y' - y.
$$

Next, for $(y,t)$, $(y',t') \in [0,1] \times [0,T]$, choose $\gamma \in \Gamma_t$ and $\gamma' \in \Gamma_t$ such that $y = y_C(\gamma,t)$ and $y = y_C(\gamma',t')$. Then, since, by definition (53), $\varphi(B,\gamma,t)$ is monotone also in $t$,

$$
\left| \int_W h(w)\mu_t(dw \times [y,1]) - \int_W h(w)\mu_t'(dw \times [y,1]) \right|
$$

$$
= \left| \int_W h(w) (\varphi(dw,\gamma,t) - \varphi(dw,\gamma',t')) \right|
$$

$$
\leq \left| \varphi(W,\gamma,t) - \varphi(W,\gamma',t') \right| + \left| \varphi(W,\gamma',t) - \varphi(W,\gamma',t') \right|
$$

$$
= \left| y_C(\gamma,t) - y_C(\gamma',t') \right| + \left| y_C(\gamma',t) - y_C(\gamma',t') \right|
$$

$$
= 2|y - y_C(\gamma',t)|
$$

$$
= 2|y_C(\gamma',t') - y_C(\gamma',t)|
$$

$$
\leq 2 \sup_{(\gamma'',t'') \in \Delta_T} \left| \frac{\partial y_C}{\partial t}(\gamma'',t'') \right| |t' - t|.
$$

Using Proposition 4, monotonicity, and Proposition 3, with (83), we have

$$
\frac{\partial y_C}{\partial t}(\gamma,t) = \int_{\gamma \geq \gamma'} \int_W w(y_C(\gamma',t),t) \frac{\partial \varphi}{\partial \gamma}(dw,\gamma',t') d\gamma'
$$

$$
\leq \int_{\gamma \in \Gamma} \int_W \|w\|_T \frac{\partial \varphi}{\partial \gamma}(dw,\gamma',t') d\gamma'
$$

$$
\leq M_W e^{2C_W t},
$$
which proves (21).

The initial conditions (22) and surjectivity for $y_C \in \Theta_T$ is contained in the definition (49) of $\Theta_T$, and the initial condition (17), for for $\mu_t$ in the definition (53) with Proposition 5.

For $t > 0$ and $B \in \mathcal{B}(W)$, (74), (53), (17), and (19) imply

$$
\mu_t(dw \times [0,1]) = \varphi(dw, (0, t), t) = \mu_0(dw \times [0,1]) = \lambda(dw),
$$

which proves (23).

For $(y, t) \in [0,1] \times [0, T]$ choose $\gamma \in \Gamma_t$ such that $y = y_C(\gamma, t)$. Then (74) and (70) imply

$$
\mu_t(W \times [y,1]) = \varphi(W, \gamma, t) = 1 - G(y_C)(\gamma, t) = 1 - y
$$

which proves (24).

To prove evolution equation (25), Proposition 5 implies, for $\gamma_t = (y_0, t_0) \in \Gamma_t$, $\mu_t(dw \times [y_C(\gamma, t), 1]) - \mu_{t_0}(dw \times [y_0, 1]) = \varphi(dw, \gamma, t) - \varphi(dw, \gamma, t_0)$,

for which Proposition 4 and (75) further imply

$$
\begin{aligned}
&\mu_t(dw \times [y_C(\gamma, t), 1]) - \mu_{t_0}(dw \times [y_0, 1]) \\
&= - \int_{t_0}^{t} \left( \int_{y_C(\gamma', s)}^{y_C(\gamma, s)} w(\gamma', s) \frac{\partial \varphi}{\partial \gamma}(dw, \gamma', s, d\gamma') ds \right) \\
&= - \int_{t_0}^{t} \left( \int_{y_C(\gamma, s)} w(z, s) \mu_s(dw \times dz) ds \right) \\
&\text{which proves (25).}
\end{aligned}
$$

\[\Box\]

5 Fixed point and existence of solution.

In Theorem 6 we assumed existence of a fixed point $\theta = y_C$ of a map $G$ defined in (70). To complete a proof of existence of a solution for Theorem 1, we prove that (70) has a fixed point. In fact, the assumptions (12) and (13) on $W$ imply that that the fixed point is unique. This is the core of the existence proof for Theorem 1, and we heavily rely on the explicit formulas (71) and (72).

Theorem 7 The map $G : \Theta \to \Theta$ in (73) has a unique fixed point $y_C \in \Theta_T$, namely, there is a unique $y_C$ which satisfies (81).

Proof. For $t \geq 0$ and $\theta$ and $\theta'$ in $\Theta_T$ define

$$
(84) \quad d(\theta', \theta, t) = \sup_{\gamma \in \Gamma_t} |\theta'(\gamma, t) - \theta(\gamma, t)|.
$$

We first accumulate basic formulas for evaluating $\omega_{\theta,w,z}$ in (51) and $\Omega_{\theta,w,z}$ in (52). In the following lemma, we write $\Omega_{\theta,w,z}(s, t)$ also for $s > 0$ in (52) whenever it becomes notationally simpler, though the quantity is actually independent of $z$ for $s > 0$. Recall the notation $\hat{\Omega}_w(s, t)$ in (61).

Lemma 8 Let $\theta \in \Theta_T$ and $\theta' \in \Theta_T$. Then for $(\gamma, t) \in \Delta_T$ with $\gamma = (z, s)$, we have

(i) $w(1, t) - C_W \leq w(\theta(\gamma, t), t) \leq w(1, t) + C_W$,

(ii) $0 < e^{-\Omega_{\theta,w,z}(s,t)} \leq e^{-\Omega_w(s,t)+C_W(t-s)}$.
Proof. The first estimate is an elementary consequence of (13) and the mean value theorem, if one notes that $\theta(\gamma, t) \in [0,1]$. This and the definitions (52) and (61), and non-negativity of $w \in W$ leads to the second estimate. With the definition (84), the third estimate is similarly proved as the first one. The last estimate follows from these estimates and

$$|e^{-x'} - e^{-x}| = |e^{-(x' + x)} - e^{-(x' \wedge x)}| = e^{-(x' \wedge x)} (1 - e^{-|x'-x|}) \leq e^{-|x'|} |x'-x|.$$ \hfill \Box

Lemma 9 It holds that

\[ d(G(\theta'), G(\theta), t) \leq 2C_W e^{2C_W T} \int_0^t d(\theta', \theta, v) dv, \quad \theta, \theta \in \Theta_T, \quad t \in [0, T]. \] \tag{85} \]

Proof. If $\gamma = (y_0, 0) \in \Gamma_i$, then applying Lemma 8 to (71), we have

$$|G(\theta')(\gamma, t) - G(\theta)(\gamma, t)| \leq C_W e^{C_W t} \int_{W \times [y_0, 1]} e^{-\tilde{\Omega}_w(0, t)} \sigma(w, z) \lambda(dw) dz \int_0^t d(\theta', \theta, v) dv, \quad \gamma \in \Gamma_i, \quad t \in [0, T].$$

Non-negativity of $\tilde{\Omega}_w$ and (18) and the fact that $\lambda$ is a probability measure further leads to

\[ \sup_{\gamma \in \Gamma_i} |G(\theta')(\gamma, t) - G(\theta)(\gamma, t)| \leq C_W e^{C_W T} \int_0^t d(\theta', \theta, v) dv, \quad t \in [0, T]. \] \tag{86} \]

The rest of the proof is for the case $\gamma = (0, t_0) \in \Gamma_i \cap \Gamma_b$. On applying Lemma 8 to each term of (72), we use an elementary equality

\[ \prod_{i=1}^n b_i - \prod_{i=1}^n a_i = \sum_{j=1}^{n-1} \left( \prod_{i=1}^{j-1} b_i \right) (b_j - a_j) \left( \prod_{i=j+1}^n a_i \right), \] \tag{87} \]

where, and in the following, we adopt a notation

$$\prod_{i=1}^0 b_i = \prod_{i=n+1}^n a_i = 1$$

to simplify the formulas. We apply (87) to the difference of (72) and its analog, with $\theta$ replaced by $\theta'$, where $b_i$’s are the factors depending on $\theta'$, and $a_i$’s the factors depending on $\theta$. We then apply the last 2 estimates in Lemma 8 to the factor of the form $b_i - a_i$, and apply the first 2 estimates to other factors. We have

\[ |G(\theta')(0, t_0) - G(\theta)(0, t_0)| \leq I_1(t_0, t) + I_2(t_0, t), \] \tag{88} \]
where
\[ I_1(t_0, t) = C_W e^{C_w t} \int_W e^{-\tilde{\Omega}_w(0,t)} \left( 1 + \sum_{k \geq 1} \int_{0 \leq u_1 \leq \ldots \leq u_k \leq t_0} \prod_{i=1}^k (w(1, u_i) + C_W) \prod_{i=1}^k du_i \right) \lambda(dw) \times \int_0^t d(\theta', \theta, v) dv, \]
and
\[ I_2(t_0, t) = C_W e^{C_w t} \int_W e^{-\tilde{\Omega}_w(0,t)} \sum_{k \geq 1} \int_{0 \leq u_1 \leq \ldots \leq u_k \leq t_0} \sum_{j=1}^k \left( d(\theta', \theta, u_j) \prod_{i=1}^k (w(1, u_i) + C_W) \prod_{i=1}^k du_i \lambda(dw) \right) \times \prod_{i; 1 \leq i \leq k, i \neq j} \lambda(dw). \]

We apply (47) to (89), to find
\[ I_1(t_0, t) = C_W e^{C_w t} \int_W e^{-\tilde{\Omega}_w(0,t)} e^{\tilde{\Omega}_w(0,t_0)} + C_W t_0 \lambda(dw) \times \int_0^t d(\theta', \theta, v) dv \]
\[ = C_W e^{C_w (t + t_0)} \int_W e^{-\tilde{\Omega}_w(t_0,t)} \lambda(dw) \int_0^t d(\theta', \theta, v) dv, \]
\[ \leq C_W e^{C_w T} \int_0^t d(\theta', \theta, v) dv. \]

To evaluate (90), we first change an integration variable \( u_j \) to \( v \) and change the order of summation for \( j \) and \( k \), to find
\[ I_2(t_0, t) = C_W e^{C_w t} \int_W e^{-\tilde{\Omega}_w(0,t)} \int_0^{t_0} d(\theta', \theta, v) \times \left( \sum_{j \geq 1} \int_{0 \leq u_1 \leq \ldots \leq u_j \leq v} \prod_{i=1}^{j-1} (w(1, u_i) + C_W) \prod_{i=1}^{j-1} du_i \right) \times \left( \sum_{k \geq j} \int_{v \leq u_{j+1} \leq \ldots \leq u_k \leq t_0} \prod_{i=j+1}^k (w(1, u_i) + C_W) \prod_{i=j+1}^k du_i \right) \lambda(dw) dv. \]

We apply (47) to the summation in \( j \) and to the summation in \( k \), to find
\[ I_2(t_0, t) = C_W e^{C_w t} \int_W e^{-\tilde{\Omega}_w(0,t)} \int_0^{t_0} d(\theta', \theta, v) e^{\tilde{\Omega}_w(0,v)} + C_W v e^{\tilde{\Omega}_w(v,t_0)} + C_W (t_0 - v) \lambda(dw) dv \]
\[ = C_W e^{C_w (t + t_0)} \int_W e^{-\tilde{\Omega}_w(t_0,t)} \lambda(dw) \times \int_0^{t_0} d(\theta', \theta, v) dv \]
\[ \leq C_W e^{C_w T} \int_0^t d(\theta', \theta, v) dv. \]

The equations (88), (91), and (92) imply
\[ \sup_{\gamma \in \Gamma_0 \cap \Gamma_t} |G(\theta)((0,t_0), t) - G(\theta)((0,t_0), t)| \leq 2C_W e^{C_w T} \int_0^t d(\theta', \theta, v) dv. \]
The equations (84), (86), and (93) finally imply (85). \( \square \)
Let us continue the proof of Theorem 7.

Define \( \theta_0 \in \Theta_T \) by

\[
\theta_0((y_0, t_0), t) = y_0, \quad ((y_0, t_0), t) \in \Delta_T,
\]

and define a sequence of flows \( \theta_k \in \Theta_T, k \in \mathbb{Z}^+ \), inductively by (94) and

\[
\theta_{k+1} = G(\theta_k), \quad k \in \mathbb{Z}^+.
\]

Lemma 9 implies

\[
d(\theta_{k+1}, \theta_k, t) = d(G(\theta_k), \theta_{k-1}, t) \leq C \int_0^t d(\theta_k, \theta_{k-1}, v) dv,
\]

where \( C = 2C \exp(2C_T) \). Iterating, we obtain estimates which, by induction, is seen to have an expression

\[
d(\theta_{k+1}, \theta_k, t) \leq C k \int_{0}^{t} \prod_{i=1}^{k} du_i = C k \int_{0}^{t} \left[ \frac{(t-u)^{k-1}}{(k-1)!} \right] \left[ \frac{1}{k!} \right] du,
\]

where \( t \in [0, T] \), \( k \in \mathbb{Z}^+ \).

Since \( \theta \in \Theta_T \) takes values in \([0, 1]\), the definition (84) implies

\[
d(\theta', \theta, t) \leq 1, \quad \theta, \theta' \in \Theta_T, \quad t \in [0, T].
\]

Substituting (98) in (97),

\[
d(\theta_{k+1}, \theta_k, t) \leq \left( \frac{C t}{k!} \right)^k, \quad k \in \mathbb{Z}^+.
\]

Since the summation in \( k \) of the right hand side of (99) converges to \( e^{C t} \),

\[
\theta_k(\gamma, t) = \theta_0(\gamma, t) + \sum_{i=0}^{k-1} \left( \theta_{i+1}(\gamma, t) - \theta_i(\gamma, t) \right)
\]

converges uniformly in \((\gamma, t) \in \Delta_T\). Denote the limit as

\[
y_C(\gamma, t) = \lim_{k \to \infty} \theta_k(\gamma, t), \quad \gamma, t \in \Delta_T.
\]

The equations (100), (95), and (99) imply (81). Since \( \theta_k \in \Theta_T \) for all \( k \), \( y_C \) also takes values in \([0, 1]\), non-decreasing in \( \gamma \) for each \( t \), and non-decreasing in \( t \) for each \( \gamma \). Since the convergence (100) is uniform in \((\gamma, t)\), and \( \theta_k \in \Theta_T \) are continuous, \( y_C \) is also continuous. Also

\[
y_C((y_0, t_0), t_0) = y_0, \quad (y_0, t_0) \in \Gamma_T.
\]

holds. In particular, \( y_C((0, t), t) = 0 \) holds, and also (50) implies \( y_C((1, 0), t) = 1 \), hence with continuity, \( y_C \) is surjective in \( \gamma \) for each \( t \). This proves \( y_C \in \Theta_T \), namely, existence of a fixed point of \( G \) in \( \Theta_T \).

Suppose there is another fixed point \( \tilde{y}_C \in \Theta_T \) of \( G \). Then (81) and Lemma 9 imply

\[
d(\theta', \theta, t) = d(G(\theta'), G(\theta), t) \leq C \int_0^t d(\theta', \theta, v) dv, \quad t \in [0, T].
\]

where \( C = 2C \exp(2C_T) \). Gronwall’s inequality implies \( d(\theta', \theta, t) = 0, \quad t \in [0, T] \). Namely, \( \theta' = \theta \). This proves uniqueness of the fixed point of \( G \). \( \square \)
6 Uniqueness of the solution.

In previous sections we proved existence of a solution \((y_C, \mu_t)\) in Theorem 1. In this section we complete the proof of Theorem 1 by proving that the solution is unique. Assume that \((y_C, \mu_t)\) and \((\tilde{y}_C, \tilde{\mu}_t)\) are the pairs which satisfy all the properties stated in Theorem 1.

Fix \(\gamma = (y_0, t_0) \in \Gamma\), and let \(t \geq t_0\). Since by assumption \((\mu_t, y_C)\) satisfy equation of motion (25) with initial and boundary conditions (22) and (23),
\[
\mu_t(dw \times [y_C(\gamma, t), 1]) = \mu_{t_0}(dw \times [y_0, 1])
+ \int_{t_0}^{t} \int_{z \in [y_C(\gamma, s), 1]} (w(1, s) - w(z, s)) \mu_s(dw \times dz) ds
- \int_{t_0}^{t} w(1, s) \mu_s(dw \times [y_C(\gamma, s), 1]) ds, \quad t \geq t_0.
\]
Note that (23) and (12) do not rule out a possibility that \(\mu_t\) has an unbounded support concerning \(\|w\|_T\). Therefore, a direct application of Gronwall type inequalities to the last term in the right hand side of (101) may lead to divergent expression upon integration with respect to \(w\). We work around this problem by the following.

**Lemma 10**
\[
\mu_t(dw \times [y_C(\gamma, t), 1]) = e^{-\tilde{\Omega}_w(t_0, t)} \mu_{t_0}(dw \times [y_0, 1])
+ \int_{t_0}^{t} e^{-\tilde{\Omega}_w(s, t)} \int_{x \in [y_C(\gamma, s), 1]} \frac{\partial w}{\partial z}(x, s) \mu_s(dw \times [y_C(\gamma, s), x]) dx ds,
\]
where \(\tilde{\Omega}_w(s, t)\) is as in (61).

**Proof.** Iterating (101) and using Fubini’s Theorem, we have
\[
\mu_t(dw \times [y_C(\gamma, t), 1])
= \mu_{t_0}(dw \times [y_0, 1]) \sum_{\ell=0}^{k} \frac{1}{\ell!} (-\tilde{\Omega}_w(t_0, t))^\ell
+ \int_{t_0}^{t} \sum_{\ell=0}^{k} \frac{1}{\ell!} (-\tilde{\Omega}_w(s, t))^\ell \int_{z \in [y_C(\gamma, s), 1]} (w(1, s) - w(z, s)) \mu_s(dw \times dz) ds
- \int_{t_0}^{t} w(1, s) \frac{1}{k!} (-\tilde{\Omega}_w(s, t))^k \mu_s(dw \times [y_C(\gamma, s), 1]) ds,
\]
\(t \geq t_0, k = 0, 1, 2, \ldots\).

Since \(w \in W\) are non-negative valued, so are \(\tilde{\Omega}_w(s, t)\) for \(s \leq t\) and
\[
0 \leq \tilde{\Omega}_w(s, t) \leq \|w\|_T, \quad 0 \leq s \leq t \leq T.
\]
Taylor’s Theorem and (104) imply
\[
\left| e^{-\tilde{\Omega}_w(s, t)} - \sum_{\ell=0}^{k} \frac{1}{\ell!} (\tilde{\Omega}_w(s, t))^\ell \right| \leq \frac{1}{(k + 1)!} (\|w\|_T)^{k+1},
0 \leq s \leq t \leq T, \quad k = 0, 1, 2, \ldots.
\]
Note also that (13) and the mean value theorem imply
\[(106)\quad |w(1, s) - w(z, s)| \leq C_T, \quad z \in [0, 1], \quad s \in [0, T].\]

Fix a constant $M > T$ (e.g., $M = 2T$), and let $B \in \mathcal{B}(W)$. Then, monotonicity of measures, (103), (104), (105), (106), and (23) imply
\[
\left| \int_B e^{-M} \|w\|_T \mu_t (dw \times [y_C(\gamma, t), 1]) - \int_B e^{-M} \|w\|_T e^{-\tilde{\Omega}(t, t)} \mu_0 (dw \times [y_0, 1]) \right| \\
- \int_0^t \int_B e^{-M} \|w\|_T e^{-\tilde{\Omega}(s, t)} \int_{z \in [y_C(\gamma, s), 1]} (w(1, s) - w(z, s)) \mu_s (dw \times dz) \, ds \\
\leq \frac{(k + 1)!}{T^{k+1}} (1 + C_B T + k + 1) \int_B e^{-M} \|w\|_T \|w\|_T^{k+1} \lambda (dw) \\
\leq \frac{(k + 1)!}{T^{k+1}} (C_B T + k + 2) \lambda (B) \sup_{x \geq 0} x^{k+1} e^{-M x} \\
\leq \sum_{\ell = 1}^{k+1} \log \ell - (k + 1) \log M \\
(107)\quad \text{By elementary calculus,} \\
\log \sup_{x \geq 0} x^{k+1} e^{-M x} = (k + 1) (\log \frac{k + 1}{M} - 1) \\
= \int_0^{k+1} \log y \, dy - (k + 1) \log M \\
\leq \sum_{\ell = 1}^{k+1} \log \ell - (k + 1) \log M = \log \left( \frac{k + 1}{M^{k+1}} \right). \\
\text{Combining (107) and (108), we have} \\
\left| \int_B e^{-M} \|w\|_T \mu_t (dw \times [y_C(\gamma, t), 1]) - \int_B e^{-M} \|w\|_T e^{-\tilde{\Omega}(t, t)} \mu_0 (dw \times [y_0, 1]) \right| \\
- \int_0^t \int_B e^{-M} \|w\|_T e^{-\tilde{\Omega}(s, t)} \int_{z \in [y_C(\gamma, s), 1]} (w(1, s) - w(z, s)) \mu_s (dw \times dz) \, ds \\
\leq \left( \frac{T}{M} \right)^{k+1} (C_B T + k + 2), \quad t \geq t_0, \quad B \in \mathcal{B}(W), \quad k = 0, 1, 2, \ldots,\]
which implies, by fixing $M > T$ and considering $k \to \infty$,
\[
\int_B e^{-M} \|w\|_T \mu_t (dw \times [y_C(\gamma, t), 1]) = \int_B e^{-M} \|w\|_T e^{-\tilde{\Omega}(t, t)} \mu_0 (dw \times [y_0, 1]) \\
+ \int_0^t \int_B e^{-M} \|w\|_T e^{-\tilde{\Omega}(s, t)} \int_{z \in [y_C(\gamma, s), 1]} (w(1, s) - w(z, s)) \mu_s (dw \times dz) \, ds \\
t \geq t_0, \quad B \in \mathcal{B}(W). \\
\text{This implies equality as a measure:} \\
\mu_t (dw \times [y_C(\gamma, t), 1]) = e^{-\tilde{\Omega}(t, t)} \mu_0 (dw \times [y_0, 1]) \\
+ \int_0^t e^{-\tilde{\Omega}(s, t)} \int_{z \in [y_C(\gamma, s), 1]} (w(1, s) - w(z, s)) \mu_s (dw \times dz) \, ds. \quad (109)
Using
\[ w(1, s) - w(z, s) = \int_z^1 \frac{\partial w}{\partial z}(x, s) \, dx \]

with Fubini’s Theorem, we arrive at (102).

Let us return to the proof of uniqueness in Theorem 1, and suppose there is another pair \((\tilde{\mu}, \tilde{\gamma}_C)\) which satisfies all the properties stated in Theorem 1. Lemma 10 implies that \((\tilde{\mu}, \tilde{\gamma}_C)\) satisfies an integral equation similar to (102),

\[
\tilde{\mu}_t(dw \times [\tilde{\gamma}_C(\gamma, t), 1]) = e^{-\tilde{\Omega}_w(t, t)} \mu_0(dw \times [y_0, 1]) + \int_{t_0}^t e^{-\tilde{\Omega}_w(s, t)} \int_{x \in [\tilde{\gamma}_C(\gamma, t), 1]} \frac{\partial w}{\partial z}(x, s) \tilde{\mu}_s(dw \times [\tilde{\gamma}_C(\gamma, s), x]) \, dx \, ds.
\]

The first term in the right hand side is equal to that of (102), because of the initial and boundary conditions (23) and (24).

Put
\[
I(t) = \sup_{h} \sup_{y \in [0, 1]} \left| \int_W h(w) \tilde{\mu}_t(dw \times [y, 1]) - \int_W h(w) \mu_t(dw \times [y, 1]) \right|
\]

and
\[
J(t) = \sup_{\gamma \in \Gamma_t} \left| \int_W h(w) \tilde{\mu}_t(dw \times [\tilde{\gamma}_C(\gamma, t), 1]) - \int_W h(w) \mu_t(dw \times [\gamma_C(\gamma, t), 1]) \right|
\]

where the supremum for \(h\) in the right hand sides of (111) and (112) are taken over measurable functions \(h : W \to \mathbb{R}\) satisfying (20). In particular, (26) implies

\[
\sup_{\gamma \in \Gamma_t} |\tilde{\gamma}_C(\gamma, t) - \gamma_C(\gamma, t)| \leq J(t).
\]

Fix \(s\) and \(x\), and put
\[
h_1(w) = \frac{1}{C_W} h(w) e^{-\tilde{\Omega}_w(s, t)} \frac{\partial w}{\partial z}(x, s).
\]

Then \(\tilde{\Omega}_w(s, t) \geq 0\) and (13) imply that \(h_1 : W \to \mathbb{R}\) satisfies (20) with \(h = h_1\). The definitions (111) and (112) then imply

\[
\left| \int_W h(w) e^{-\tilde{\Omega}_w(s, t)} \frac{\partial w}{\partial z}(x, s) \tilde{\mu}_s(dw \times [\tilde{\gamma}_C(\gamma, s), x]) \right|
\]

\[
\leq C_W \left| \int_W h_1(w) \tilde{\mu}_s(dw \times [\tilde{\gamma}_C(\gamma, s), 1]) - \int_W h_1(w) \mu_s(dw \times [\gamma_C(\gamma, s), 1]) \right|
\]

\[
+ C_W \left| \int_W h_1(w) \tilde{\mu}_s(dw \times [x, 1]) - \int_W h_1(w) \mu_s(dw \times [x, 1]) \right|
\]

\[
\leq C_W (I(s) + J(s)).
\]

Also, monotonicity of measure implies

\[
\left| \int_W h(w) e^{-\tilde{\Omega}_w(s, t)} \frac{\partial w}{\partial z}(x, s) \tilde{\mu}_s(dw \times [\tilde{\gamma}_C(\gamma, s), x]) \right|
\]

\[
\leq C_W \tilde{\mu}_s(W \times [0, 1]) = C_W.
\]
Substituting (102) and (110) in (112), using (114) and (115), and also (113), we have
\[
J(t) \leq \sup_{h} \sup_{\gamma \in \Gamma} \int_{t_0}^{t} \left( \int_{y_C(\gamma, s)}^{y_C(\gamma, s)} C_W \, dx \right) \left( \int_{x \in [y_C(\gamma, s), 1]} C_W \left( I(s) + J(s) \right) \, dx \right) \, ds \\
\leq C_W \sup_{h} \sup_{\gamma \in \Gamma} \int_{t_0}^{t} \left( \left| y_C(\gamma, s) - y_C(\gamma, s) \right| + I(s) + J(s) \right) \, ds \\
\leq \int_{0}^{t} \left( I(s) + 2J(s) \right) \, ds.
\]

Next, since by assumption, for each \( t \), the map \( \gamma \mapsto y_C(\gamma, t) \) is surjective, we have, from (111),
\[
I(t) = \sup_{h} \sup_{\gamma \in \Gamma} \left| \int_{W} h(w) \mu_t(dw \times [y_C(\gamma, t), 1]) - \int_{W} h(w) \mu_t(dw \times [y_C(\gamma, t), 1]) \right|.
\]
Using Lipschitz continuity (21), the definition (112), and (113), we have
\[
I(t) \leq \sup_{h} \sup_{\gamma \in \Gamma} (|\bar{y}_C(\gamma, t) - y_C(\gamma, t)| + J(t)) \leq 2J(t).
\]
The inequalities (117) and (118) imply
\[
J(t) \leq 4 \int_{0}^{t} J(s) \, ds, \quad t \in [0, T],
\]
which, by Gronwall’s inequality, further implies \( J(t) = 0, \ t \in [0, T] \). This proves \( \bar{y}_C = y_C \), and also (118) now implies \( I(t) = 0, \ t \in [0, T] \), which proves \( \bar{\mu}_t = \mu_t \). This completes a proof of the uniqueness claim in Theorem 1.

**A Application of Schauder’s fixed point theorem.**

In this section we consider the case where we keep the fundamental condition (12), but replace the global Lipschitz type condition (13) by a global bound condition on oscillation:
\[
C_W := \sup_{w \in W} \sup_{(y, t), (y', t') \in [0, 1] \times [0, T]} |w(y, t) - w(y', t')| < \infty,
\]
and consider the existence of fixed points to the map \( G : \Theta_T \rightarrow \Theta_T \) defined by the explicit formula (71) and (72), where the notations are introduced in (18), (19), (14), (15), (16), (49), and (52). Note that \( G(\theta) \in \Theta_T \) holds with the weaker assumption (119). This can be shown directly from the explicit expression (72). The only condition for \( \Theta_T \) perhaps not obvious from the expression is the range condition \( G(\theta)(\gamma, t) \in [0, 1] \), which can be shown as follows. For \( k = 1, 2, \ldots \) put
\[
I_k = \int_{0 \leq u_1 \leq \ldots \leq u_k \leq t_0} w(\theta((z, 0), u_1), u_1) e^{-\Omega_{\theta, w}(0, u_1)} \\
\times \prod_{i=2}^{k} \left( w(\theta((0, u_{i-1}), u_i), u_i) e^{-\Omega_{\theta, w}(u_{i-1}, u_i)} \right) \\
\times e^{-\Omega_{\theta, w}(u_k, t)} \prod_{i=1}^{k} du_i.
\]
and
\[
J_k = \int_{0 \leq u_1 \leq \cdots \leq u_k \leq t_0} w(\theta((z, 0), u_1), u_1) e^{-\Omega_{\theta,w,z}(0,u_1)} \times \prod_{i=2}^{k} \left( w(\theta((0, u_{i-1}), u_i), u_i) e^{-\Omega_{\theta,w,z}(u_{i-1},u_i)} \right) \prod_{i=1}^{k} du_i.
\]
Note that non-negativity of \(\Omega_{\theta,w,z}\) implies \(I_k \leq J_k\). We can perform the \(u_k\) integration in \(J_k\) to find \(J_k = J_{k-1} - I_{k-1}\), which we can iterate to find
\[
\sum_{i=1}^{k} I_i = J_1 - J_k + I_k \leq J_1 = 1 - e^{-\Omega_{\theta,w,z}(0,t_0)}.
\]
Substituting this in (72), and using monotonicity of \(\Omega_{\theta,w,z}\), (19), and \(\lambda(W) = 1\), we have \(1 \geq G(\theta)((0,t_0),t) \geq 0\). A similar estimate for (71) is straightforward, hence we conclude \(G(\theta) \in \Theta_T\).

**Theorem 11**. Under the condition (12) and (119), the map \(G : \Theta_T \to \Theta_T\) has a fixed point. 

**Remark**. Since the proof relies on Schauder’s fixed point theorem, our proof has no control of uniqueness of fixed points.

The map \(G\) maps \(\Theta_T\) into itself, and \(\Theta_T\) is a subset of a Banach space (with the supremum norm) of continuous functions \(C^0(\Delta_T; [0,1])\) taking values in a finite interval \([0,1] \subset \mathbb{R}\). The domain \(\Delta_T\) is homeomorphic to a rectangle, since its parameterization in the definition (16) is homeomorphic to a trapezoid.

The Schauder fixed point theorem states [1, 2.4.3] that a compact map of a closed bounded convex set in a Banach space into itself has a fixed point. (The notational correspondence between here and [1, §2.4] is given by \(X = C^0(\Delta_T; [0,1]), K = U = \Theta_T,\) and \(f = G\).) We have shown \(G(\Theta_T) \subset \Theta_T\) at the beginning of this section.

Concerning the required properties for the domain \(\Theta_T\) of the map \(G\), we have noted that \(C^0(\Delta_T; [0,1])\) is a bounded set. For a sequence of continuous and monotone functions, the limit function with respect to the supremum norm also is continuous and monotone, and since for \(\theta \in \Theta_T\), \(\theta((0, t), t) = 0\) and \(\theta((1, 0), t) = 1\) holds, these properties are also preserved in the limit. This and continuity imply surjectivity of the limit function. Therefore \(\Theta_T\) is a closed set. The continuity, monotonicity, the properties \(\theta((0, t), t) = 0\) and \(\theta((1, 0), t) = 1\) are also preserved by convex linear combination, hence \(\Theta_T\) is also convex. Thus \(\Theta_T\) is a closed, bounded, convex set.

It remains to prove compactness of \(G\). Since \(C^0(\Delta_T; [0,1])\) is a bounded set with respect to the supremum norm, the Arzela-Ascoli theorem implies that it is sufficient to prove (i) that the map \(G : \Theta_T \to \Theta_T\) is continuous, and (ii) that the functions in the image set \(G(\Theta_T)\) are equicontinuous, which we prove in Lemma 13 and Lemma 14, respectively.

Note first that non-negativity of \(w \in W\) obviously implies
\[
0 < e^{-\Omega_{w}(s,t)} \leq 1, \ s \leq t,
\]
where \(\tilde{\Omega}_{w}(s,t)\) is as in (61).

**Proposition 12** For \(\theta\) and \(\theta'\) in \(\Theta_T\), and \((\gamma, t) \in \Delta_T\) with \(\gamma = (z, s)\), we have
\[
(i) \ w(1,t) - C'_W \leq w(\theta(\gamma, t), t) \leq w(1,t) + C'_W \leq ||w||_T + C'_W, \\
(ii) \ 0 < e^{-\Omega_{\theta,w,z}(s,t)} \leq e^{-\tilde{\Omega}_{w}(s,t)+C'_W (t-s)},
\]
Proof. (119) implies
\[
|w(y, t) - w(1, t)| \leq C_W', \quad w \in W; \quad (y, t) \in [0, 1] \times [0, T],
\]
which further implies
\[
0 < e^{-\Omega_{y, w}(s, t)} \leq e^{-\int_s^t w(\theta'(\gamma, u)) \, du + C_W'(t-s)},
\]
These estimates imply the first 2 estimates. The last estimate follows from these estimates and
\[
|e^{-x'} - e^{-x}| = |e^{-(x' \wedge x)} - e^{-(x' \wedge x)}| = e^{-(x' \wedge x)} (1 - e^{-|x'-x|}) \leq e^{-(x' \wedge x)} |x' - x|.
\]

Lemma 13 \( G : \Theta_T \to \Theta_T \) is a continuous map.

Proof. Let \( \theta, \theta' \in \Theta_T \), and put \( (\gamma, t) \in \Delta_T \) and \( \gamma = (y_0, t_0) \).

If \( \gamma = (y_0, 0) \in \Gamma, \quad (t_0 = 0), \quad (71), \quad \text{Proposition 12}, \quad (120), \quad \text{and} \quad (19) \) imply
\[
\sup_{y_0 \in [0, 1]} \sup_{t \in [0,T]} |G(\theta')(y_0, 0, t) - G(\theta)(y_0, 0, t)|
\]
\[
\leq \sup_{y_0 \in [0, 1]} \sup_{t \in [0,T]} \int_{W \times \{y_0, 1\}} e^{-\tilde{\Omega}_{w}(0, t)} + C_W' \int_0^t |w(\theta'(z, 0, u)) - w(\theta(z, 0, u))| \, du
\]
\[
\times \sigma(w, z) \lambda(dw) \, dz
\]
\[
\leq T e^{C_W' T} \sup_{W \times \{0, 1\}} \sup_{u \in [0,T]} |w(\theta'(z, 0, u)) - w(\theta(z, 0, u))| \lambda(dw).
\]
Concerning the rightmost hand side, we have
\[
\sup_{u \in [0,T]} \sup_{z \in [0,1]} |w(\theta'(z, 0, u)) - w(\theta(z, 0, u))| \leq 2 \|w\|_T,
\]
while (12) implies \( \int_{W \times \{0, 1\}} \|w\|_T \lambda(dw) = M_W < \infty \). Hence the integrand in the right hand side of (121) is bounded, pointwise in \( w \in W \), uniformly in \( \theta' \) by an integrable function. Therefore, thanks to dominated convergence theorem we may interchange the order of integration and the limit \( \theta' \to \theta \) in the right hand side of (121).

\( W \subset C^4([0, 1] \times [0, T]; [0, \infty]) \) and \( [0, 1] \times [0, T] \) is compact, hence each \( w \in W \) is uniformly continuous. Hence for any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that
\[
(\forall y, y' \in [0, 1] ; |y' - y| < \delta)(\forall u \in [0, T]) |w(y, u) - w(y', u)| < \epsilon.
\]
If the supremum norm of \( \theta' - \theta \) is less than \( \delta \) we have
\[
\sup_{u \in [0,T]} \sup_{z \in [0,1]} |\theta'(z, 0, u) - \theta(z, 0, u)| < \delta,
\]
which further implies
\[
\sup_{u \in [0, T]} \sup_{z \in [0, 1]} |w(\theta'((z, 0), u), u) - w(\theta((z, 0), u), u)| \leq \epsilon,
\]
hence (121) implies
\[
\lim_{\theta' \to \theta} \sup_{y_0 \in [0, 1]} \sup_{t \in [0, T]} |G(\theta')((y_0, 0), t) - G(\theta)((y_0, 0), t)| \leq T e^{C_W T} \epsilon.
\]
Since \( \epsilon > 0 \) is arbitrary,
\[
(122) \quad \lim_{\theta' \to \theta} \sup_{y_0 \in [0, 1]} \sup_{t \in [0, T]} |G(\theta')((y_0, 0), t) - G(\theta)((y_0, 0), t)| = 0.
\]
Next if \( \gamma = (0, t_0) \in \Gamma_t \cap \Gamma_b \) \( (y_0 = 0) \), we proceed as in exact analogy to the proof of Lemma 9, to find
\[
\sup_{t \in [0, T]} \sup_{t_0 \in [0, t]} |G(\theta')((0, t_0), t) - G(\theta)((0, t_0), t)|
\leq T e^{2C_W T} \int_{W} \sup_{\gamma \in \Delta_T} |w(\theta'((\gamma, u), u) - w(\theta(\gamma, u), u)| \lambda(dw).
\]
Therefore, as in the same reasoning as we derive (122) from (121),
\[
(123) \quad \lim_{\theta' \to \theta} \sup_{t \in [0, T]} \sup_{t_0 \in [0, t]} |G(\theta')((0, t_0), t) - G(\theta)((0, t_0), t)| = 0.
\]
Finally, (122) and (123) imply
\[
\lim_{\theta' \to \theta} \sup_{(\gamma, t) \in \Delta_T} |G(\theta')(\gamma, t) - G(\theta)(\gamma, t)| = 0,
\]
which proves the continuity of \( G : \Theta \to \Theta \).

\[ \square \]

**Lemma 14** The functions in the set \( G(\Theta_T) \) are equicontinuous.

**Proof.** We see from elementary calculus using the mean value theorem and triangular inequality that the following uniform estimates on the derivatives of \( G(\theta) \) imply equicontinuity:
\[
0 \leq \frac{\partial}{\partial y_0} G(\theta)((y_0, 0), t) \leq 1, \quad y_0 \in [0, 1], \ t \in [0, T],
\]
\[
0 \leq \frac{\partial}{\partial t} G(\theta)((y_0, 0), t) \leq M_W + C_W', \quad y_0 \in [0, 1], \ t \in [0, T],
\]
\[
0 \leq -\frac{\partial}{\partial t_0} G(\theta)((0, t_0), t) \leq (M_W + C_W') e^{2C_W T}, \quad t_0 \in [0, t], \ t \in [0, T],
\]
\[
0 \leq \frac{\partial}{\partial t} G(\theta)((0, t_0), t) \leq (M_W + C_W') e^{2C_W T}, \quad t_0 \in [0, t], \ t \in [0, T].
\]
The remainder of the proof is devoted to proving these estimates.

To prove (124), differentiate the explicit formula (71) by \( t_0 \) and use (120) and (18). We have
\[
0 \leq \frac{\partial}{\partial y_0} G(\theta)((y_0, 0), t) = \int_{W} e^{-\Omega_{\theta, w}(0, t)} \sigma(w, y_0) \lambda(dw)
\leq \int_{W} \sigma(w, y_0) \lambda(dw) = 1,
\]
which proves (124).

To prove (125), differentiate (71) by $t$, and use (52), Proposition 12, (120), (19), and (12), to find
\[
0 \leq \frac{\partial}{\partial t} G(\theta)((y_0, 0), t) = \int_{W \times [y_0, 1)} w(\theta((z, 0), t), t) e^{-\Omega_{\theta,w,z}(0,t)} \sigma(w, z) \lambda(dw) dz
\]
\[
\leq \int_{W \times [0,1)} (\|w\|_T + C'_W) \sigma(w, z) \lambda(dw) dz = M_W + C'_W,
\]
which proves (125).

Proofs of (126) and (127) are similar. We differentiate (72) by $t_0$ and $t$, respectively, and follow a similar line. The only new point is that we apply (47) in a similar way as in the proof of Lemma 9. This completes a proof of Lemma 14. \hfill \Box

As discussed in the beginning of this appendix, Lemma 13 and Lemma 14 prove Theorem 11.

References


