Restoration of isotropy on fractals.

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Abstract

We report a new type of restoration of *macroscopic isotropy (homogenization)* in fractals with *microscopic anisotropy*. The phenomenon is observed in various physical setups, including diffusions, random walks, resistor networks, and Gaussian field theories. The mechanism is unique in that it is absent in uniform media, while universal in that it is observed in a wide class of fractals.

In this letter, we report a new type of restoration of macroscopic isotropy (homogenization) in fractals with microscopic anisotropy. The phenomenon is unique in that it is absent in uniform media, while universal in that it is observed in a wide class of fractals. We suspect that the phenomenon is universal enough to be observed experimentally, for example, in spin systems close to critical points and various transport phenomena in fractal media. We first discuss the Sierpiński gasket as an example of finitely ramified fractals, where the calculations can be performed explicitly. We then turn to the Sierpiński carpet, an infinitely ramified fractal, and report on rigorous results. We conclude by discussing an intuitive picture of the mechanism. Some results of numerical calculations are also presented.

We note that when we discuss 'isotropy' for a deterministic regular fractal, we mean invariance with respect to (discrete) rotations which respect the structure of the fractal.

Resistor network on Sierpiński gasket. In order to illustrate the phenomenon of isotropy restoration, we first concentrate on the simplest example of anisotropic resistor network on the Sierpiński gasket, a typical finitely ramified fractal. Let n be a non-negative integer, and put $O = (0,0)$, $a_n = (2^n, 0)$, and $b_n = (2^{n-1}, 2^{n-1}\sqrt{3})$. Consider the n-th generation of the (pre-)Sierpiński gasket, which is a triangle $\triangle O a_n b_n$ with self-similar internal structure composed of triangles of side length 1, as illustrated in (Fig. 1). Each internal vertex has 4 bonds of unit length attached. We associate a resistor of resistance 1 with each bond parallel to the x-axis, and a resistor of resistance $r > 1$ with the remaining bonds. By repeated use of the star–triangle relations (Y–Δ transforms), this n-th level network can be reduced to a simple triangular network (an effective network), with resistances $R^x_n(r)$ in the horizontal bond Oa_n and $R^y_n(r)$ in the bonds Ob_n and $a_n b_n$. By definition, $R^x_0(r) = 1$ and $R^y_0(r) = r$. Put

$$H_n(r) = R^y_n(r)/R^x_n(r)\,. \tag{1}$$

$H_n(r)$ measures the effective anisotropy of $\triangle O a_n b_n$ composed of resistance elements with the basic (microscopic) anisotropy parametrized by $r = H_0(r)$. Using the star–triangle relations we obtain the following recursion relations for R^x_n and R^y_n:

$$R^x_{n+1} = \frac{2R^x_n R^y_n (2R^x_n + 3R^y_n)(3R^x_n + 2R^y_n)}{({R^x_n}^2 + 6R^x_n R^y_n + 3{R^y_n}^2)(R^x_n + 2R^y_n)}\,,$$
$$R^y_{n+1} = \frac{R^y_n (2R^x_n + 3R^y_n)}{R^x_n + 2R^y_n}\,.$$

We see from these formula that in the anisotropic regime ($H_n(r) \gg 1$), the effective resistances satisfy the scaling behavior

$$R^x_{n+1}(r) \approx 2R^x_n(r)\,, \quad R^y_{n+1}(r) \approx (3/2)R^y_n(r)\,, \tag{2}$$

while in the isotropic regime ($H_n(r) \approx 1$), we have

$$R^x_{n+1}(r) \approx R^y_{n+1}(r) \approx (5/3)R^x_n(r)\,. \tag{3}$$

We also see that $H_n(r)$ in (1) satisfies $H_{n+1}(r)^{-1} = f(H_n(r)^{-1})$, where

$$f(x) = (4x + 6x^2)/(3 + 6x + x^2)\,. \tag{4}$$

In particular, we see the restoration of isotropy,

$$\lim_{n\to\infty} H_n(r) = 1\,. \tag{5}$$

Fig. 2 gives the calculated behaviors of the effective resistances. We see a clear signal of the two scaling regimes (2) and (3). Using (4), we can calculate the rates of restoration of isotropy. In the anisotropic regime, we have $H_{n+1}(r) \approx (3/4)H_n(r)$, while in the isotropic regime, we have $H_{n+1}(r)-1 \approx (4/5)(H_n(r)-1)$. We can also calculate the scaling limit $F(z) = \lim_{n\to\infty} f^n((3/4)^n z) = z - (3/2)z^2 + (39/14)z^3 + \cdots$, where f^n is the n-th iteration of f. For large r and large n ($1 \ll n < O(\log(r)/\log(4/3))$) we have $H_n^{-1}(r) \approx F((4/3)^n/r)$. We can prove by standard methods using (4) that the scaling limit exists and that F is complex analytic in a neighborhood of $z = 0$.

We can generalize the above consideration so that the resistors parallel to Ob_0 and a_0b_0 have different values. If we denote the effective resistances parallel to Oa_0, Ob_0, a_0b_0, by R^a_n, R^b_n, R^c_n, respectively, we find $R^a_{n+1} = \frac{(4K+R^a_n+R^b_n+R^c_n)R^a_n(R^b_n+R^c_n)}{(K+R^b_n+R^c_n)(R^a_n+R^b_n+R^c_n)}$, where $K = \frac{(R^a_n+R^b_n)(R^b_n+R^c_n)(R^c_n+R^a_n)}{2(R^a_nR^b_n+R^b_nR^c_n+R^c_nR^a_n)}$. Corresponding formula for R^b_{n+1} and R^c_{n+1} are obtained by cyclic permutations of the suffixes. Restoration of isotropy $\lim_{n\to\infty} R^b_n/R^a_n = \lim_{n\to\infty} R^c_n/R^a_n = 1$ can be proved in the generalized situation.

Restoration of isotropy is not observed in uniform media. To see this, consider a resistor network of regular square lattice, whose horizontal (resp. vertical) bonds are resistors of resistance 1 (resp. r). The ratio of the effective resistances for $n \times n$ size network in vertical direction to horizontal direction is easily seen to be r, independently of n. Thus the anisotropy for the resistor network of regular lattice is independent of scale. The restoration of isotropy which we observe on the Sierpiński gasket is a feature absent on uniform media.

Related models on Sierpiński gasket. We described restoration of isotropy in terms of resistor networks [1, 2]. The phenomenon is also observed in various other physical setups, including random walks and diffusions [3, 4] and Gaussian field theories [5]. A related mathematical problem of the construction and uniqueness of diffusions on the Sierpiński gasket is dealt with in [6]. We also remark that there is another aspect in homogenization, that a diffusion with microscopic irregularity restores macroscopic uniformity, as studied in [7] for finitely ramified fractals. This aspect, in contrast to what we deal with here, is not specific to fractals and has been known in Euclidean spaces. (For other related references in mathematics literature, see the references in [8].)

Restoration of isotropy on Sierpiński carpet. The finite ramifiedness of the Sierpiński gasket implies that the recursion relations are finite dimensional, and the analysis can be made explicitly. One might then wonder if the isotropy restoration we found above is a special feature of models on finitely ramified fractals. In [9] we have proved a mathematical theorem for a class of infinitely ramified fractals, which establishes that the isotropy restoration is a universal phenomenon.

To state the result of [9], let n be a non-negative integer, and consider the pre-Sierpiński carpet F_n, which is a subset of a unit square $[0, 1] \times [0, 1]$ obtained by removing small squares recursively as for constructing the Sierpiński carpet [10], until squares of side length 3^{-n} are reached, where we stop so that smaller scale structures are absent (Fig. 3). Let $r > 1$, and assume that F_n is made of a material with a uniform but anisotropic electrical resistivity, such that for a unit square made of this material, the total resistance is 1 in the x-direction and r in the y-direction, and the principal axes of the resistivity tensor are parallel to the x and y axes. Equivalently, we assume that the energy dissipation rate per unit area for the potential (voltage) distribution $v(x, y)$ is $(\frac{\partial v}{\partial x})^2 + \frac{1}{r}(\frac{\partial v}{\partial y})^2$. (Note that by linear transform in coordinate $y' = y\sqrt{r}$, the formula becomes that of isotropic material. Hence, in experimental situation, one may as well start with a rectangle made of isotropic material, with rectangular holes.)

We introduce the effective resistance $R_n^x(r)$ of F_n in x direction, the resistance observed when we apply voltage between two edges $x = 0$ and $x = 1$. Likewise we define $R_n^y(r)$ and introduce the effective anisotropy $H_n(r)$, as in (1). $H_0(r) = r$ parametrizes the anisotropy of the material composing F_n. We can prove the following [9].

Theorem 1 — There is a finite constant $C \geq 1$, independent of r and n, such that for any initial anisotropy $r > 0$, we have the weak restoration of isotropy (weak homogenization) in the sense that $1/C \leq H_n(r) \leq C$ holds for sufficiently large n. (How large n should be depends on the value of r.)

We believe that C can be taken arbitrarily close to 1, as in (5), but this is still beyond the reach of present mathematical techniques, for the infinitely ramified fractals. We emphasize that we have concrete rigorous results as Theorem 1, in spite of the difficulties for the infinitely ramified fractals.

Analogous results hold if we consider a cross-wire network G_n defined by replacing each smallest size square of F_n by a horizontal and vertical cross-wire of four resistors (connected at the center of the square), whose resistances are $1/2$ in horizontal direction and $r/2$ in the vertical direction. The results stated above for the board F_n also hold for the network G_n.

Ideas for a proof of the Theorem. Theorem 1 is proved by decomposing the problem into the isotropic regime and the anisotropic regime. For the isotropic regime, an extension (to anisotropic case) of a deep renormalization group-type analysis of effective resistance for the isotropic Sierpiński carpet [2, 11] is applied, while for the anisotropic case, renormalization group-type picture in the neighborhood of degenerate fixed points [5, 3, 4] holds. One of the key observations for the proof of Theorem 1 is that if $H_n(r)$ is very large (in the anisotropic regime), then $H_n(r)$ follows a scaling behavior. We can prove

Theorem 2 — The limits $\lim_{s\to\infty} s^{-1} \liminf_{n\to\infty} H_n((9/7)^n s) = \lim_{s\to\infty} s^{-1} \limsup_{n\to\infty} H_n((9/7)^n s)$ exist.

This result says that while $s = (7/9)^n r$ and n are large, $H_n(r)$ decreases like $c\,(7/9)^n r$. We can prove these Theorems by giving bounds controlling the n dependence of the effective resistances [9]. Roughly speaking, we can show that in the anisotropic regime ($H_n(r) \gg 1$),

$$R_{n+1}^x(r) \approx (3/2)\, R_n^x(r)\,, \quad R_{n+1}^y(r) \approx (7/6)\, R_n^y(r)\,, \tag{6}$$

while in the isotropic regime ($H_n(r) \approx 1$), $R_{n+1}^x(r) \approx \rho\, R_n^x(r)$, and $R_{n+1}^y(r) \approx \rho\, R_n^y(r)$. Here $\rho = 1.25148 \pm 1 \times 10^{-5}$ is the growth exponent for the effective resistance in the isotropic case $r = 1$ [2, 11].

Based on these results, we conjecture that (5) holds also for the Sierpiński carpet, and that Fig. 2 schematically gives the behaviors of $R_n^x(r)$ and $R_n^y(r)$.

Discussions. Our mathematical results are not very sharp numerically; we can only say that $10^{-10} < H_n(r) < 10^{10}$, for large n. Numerical calculations for the Sierpiński carpet may therefore be of interest. We give results for the resistor network G_n. Obviously, $R_0^x(r) = 1$ and $R_0^y(r) = r$. It is not difficult to find $R_1^x(r) = (3r + 4)/(2r + 3)$. The exact result for $n = 2$ is

$$R_2^x(r) = \frac{324r^8 + 3960r^7 + 17169r^6 + 37077r^5 + 44639r^4 + 30842r^3 + 11900r^2 + 2325r + 174}{144r^8 + 1924r^7 + 8850r^6 + 20052r^5 + 25146r^4 + 17976r^3 + 7128r^2 + 1422r + 108}\,.$$

Note that $R_n^y(r) = r\, R_n^x(1/r)$, with which we can calculate $R_n^y(r)$ and $H_n(r)$ from these formula. We have numerical results for $3 \leq n \leq 7$, obtained using Gaussian relaxation method (Table 1). We see that as r is increased, the n dependence of $R_n^x(r)$ rapidly approach $(3/2)^n$, and that for large n, those of $R_n^y(r)$ approach $c\,(7/6)^n$ with $c = 6/5$. These observations are consistent with (6), implying scaling behavior in the anisotropic regime. (Deviation from scaling of R_n^y for small n in the data can be explained if we notice that we are calculating the network G_n instead of the board F_n.) In particular, we see that for any value of $r > 1$, R_n^y/R_n^x monotonically decreases as n is increased, which indicates the tendency of restoration of isotropy.

We expect that the scaling limit

$$z \lim_{n\to\infty} H_n((9/7)^n/z) = c + d\,z + \cdots$$

exists, where c is the limit in Theorem 2. The data and the fact that $R_n^x(r)$ is a rational function of r makes it possible to find an estimate

$$R_n^x(0) = \lim_{r\to\infty} r^{-1} R_n^y(r) = c\,(7/6)^n - 3^{-n}/5$$

with $c = 6/5$. Thus the constant term c in the scaling function is determined. We need more data to determine d, but the calculations become rapidly time consuming as n or r is increased.

Let us discuss general intuitive picture of the restoration of isotropy, in terms of random walks [3, 4]. The fractals may be regarded to have obstacles or holes in the space, when compared to uniform spaces. Intuitively, a random walker that favors horizontal motion performs a one-dimensional random walk between a pair of obstacles, and eventually is forced to move in off-horizontal direction before he could move further horizontally. There are obstacles of various sizes, separated by distances of the same order as their sizes, hence globally, the random walker is scattered almost isotropically. On uniform media such as regular lattices or Euclidean spaces, these obstacles are absent, hence the anisotropic walk keeps anisotropy asymptotically.

The Sierpiński gasket and the Sierpiński carpet have exact self-similarity, and one may doubt the 'extrapolation' to figures without exact self-similarity. However, we can prove that the restoration of isotropy occurs for anisotropic diffusions on the scale-irregular *abb*-gaskets, a family of fractals which are scale-irregular, i.e. do not have exact self-similarity [4]. These considerations suggest that the restoration of isotropy is to be observed on a wide class of random media. For example, numerical calculations on the percolation clusters may provide interesting observations.

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References

[1] H. Watanabe, J. Phys. A **18**, 2807 (1985).

[2] M. T. Barlow, R. F. Bass, Proc. Roy. Soc. London A **431**, 345 (1990).

[3] K. Hattori, T. Hattori, H. Watanabe, Probab. Theory Relat. Fields **100**, 85 (1994).

[4] T. Hattori, *Asymptotically one-dimensional diffusions on scale-irregular gaskets,* preprint (1994).

[5] K. Hattori, T. Hattori, H. Watanabe, Progr. Theor. Phys. Supplement **92**, 108 (1987).

[6] H. Osada, *Self–similar diffusions on finitely–ramified fractals,* preprint (1995).

[7] T. Kumagai, S. Kusuoka, *Homogenization on nested fractals,* preprint (1995).

[8] T. Hattori, H. Nakajima, *Transition density of diffusion on the Sierpiński gasket and extension of Flory's formula,* Phys. Rev. E, to appear.

[9] M. T. Barlow, K. Hattori, T. Hattori, H. Watanabe, *Weak homogenization of anisotropic diffusion on pre-Sierpiński carpet,* preprint (1995).

[10] W. Sierpiński, C. r. hebd. Śeanc. Acad. Sci., Paris **162**, 629 (1916).

[11] M. T. Barlow, R. F. Bass, J. D. Sherwood, Proc. Roy. Soc. London A **431**, 345 (1990).

Figure 1: Pre-Sierpiński gasket.

Figure 2: $R_n^x(r)$ (lower plots) and $R_n^y(r)$ (upper plots) on the pre-Sierpiński gasket for $r = 100$. The lines are the scaling predictions (2) and (3).

Figure 3: Pre-Sierpiński carpet F_3.

Table 1: Effective resistances $R_n^x(r)$ and $R_n^y(r)$ for the pre-Sierpiński carpet network G_n.

	$r = 10$		$r = 100$		$r = 1000$		$r = 10000$		$r = 100000$	
n	$R_n^x(r)$	$R_n^y(r)$	$R_n^x(r)$	$R_n^y(r)$	$R_n^x(r)$	$R_n^y(r)$	$R_n^x(r)$	$R_n^y(r)$	$R_n^x(r)$	$R_n^y(r)$
3	2.831057	19.64149	3.238145	190.6445	3.356806	1899.017	3.373110	18982.35	3.374810	189815.7
4	3.798415	23.47825	4.614455	224.0274	4.963201	2223.085	5.049858	22209.29	5.061194	222070.3
5	5.070868	28.10055	6.524220	263.1750	7.258880	2598.702	7.524180	25934.86	7.585124	
6	6.742934	33.69136	9.185975	309.3891	10.56635	3037.488	11.14879	30272.61	11.34244	
7	8.933314	40.46672	12.88375	364.0724	15.34037					

Fig. 1

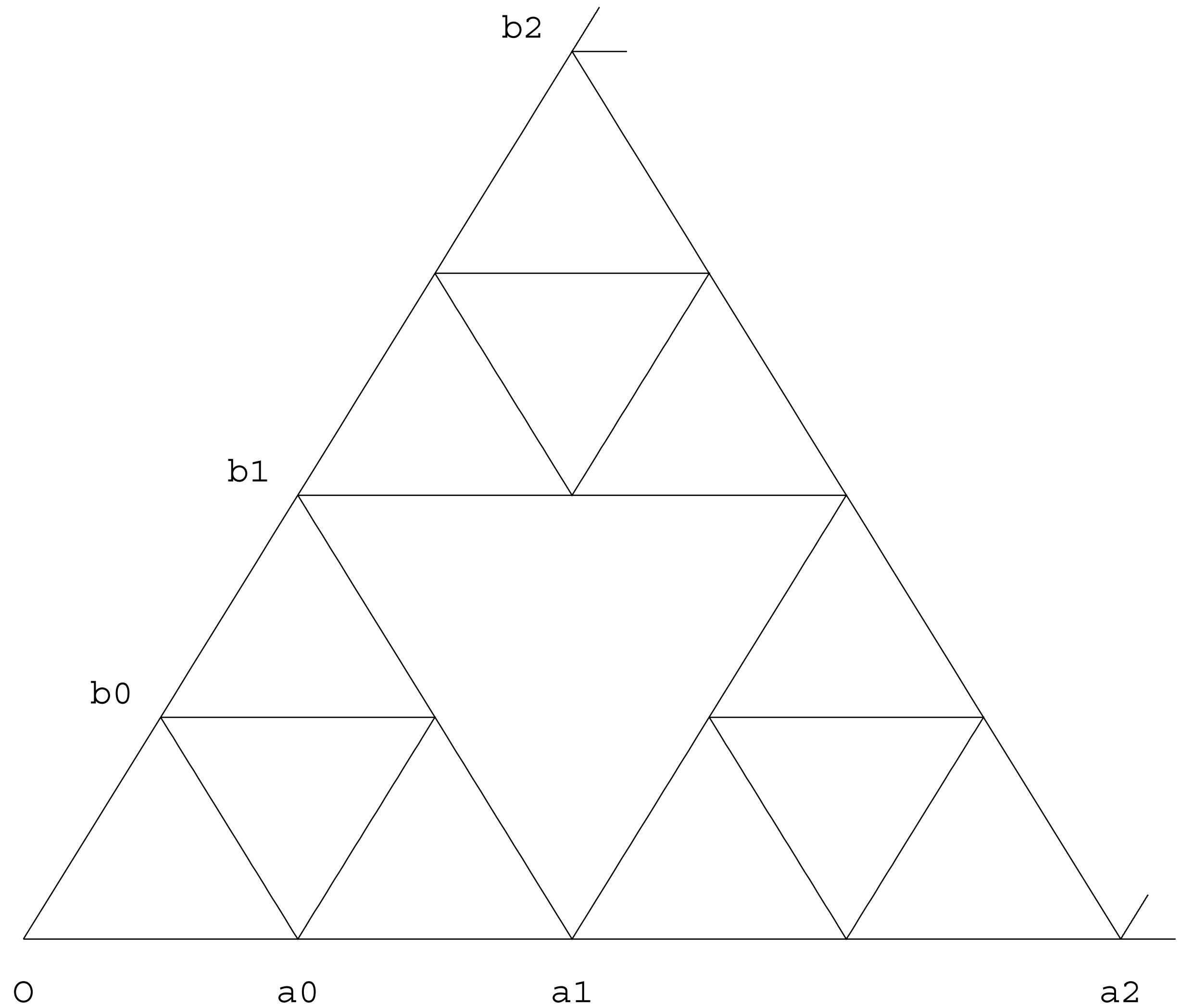

Fig. 2

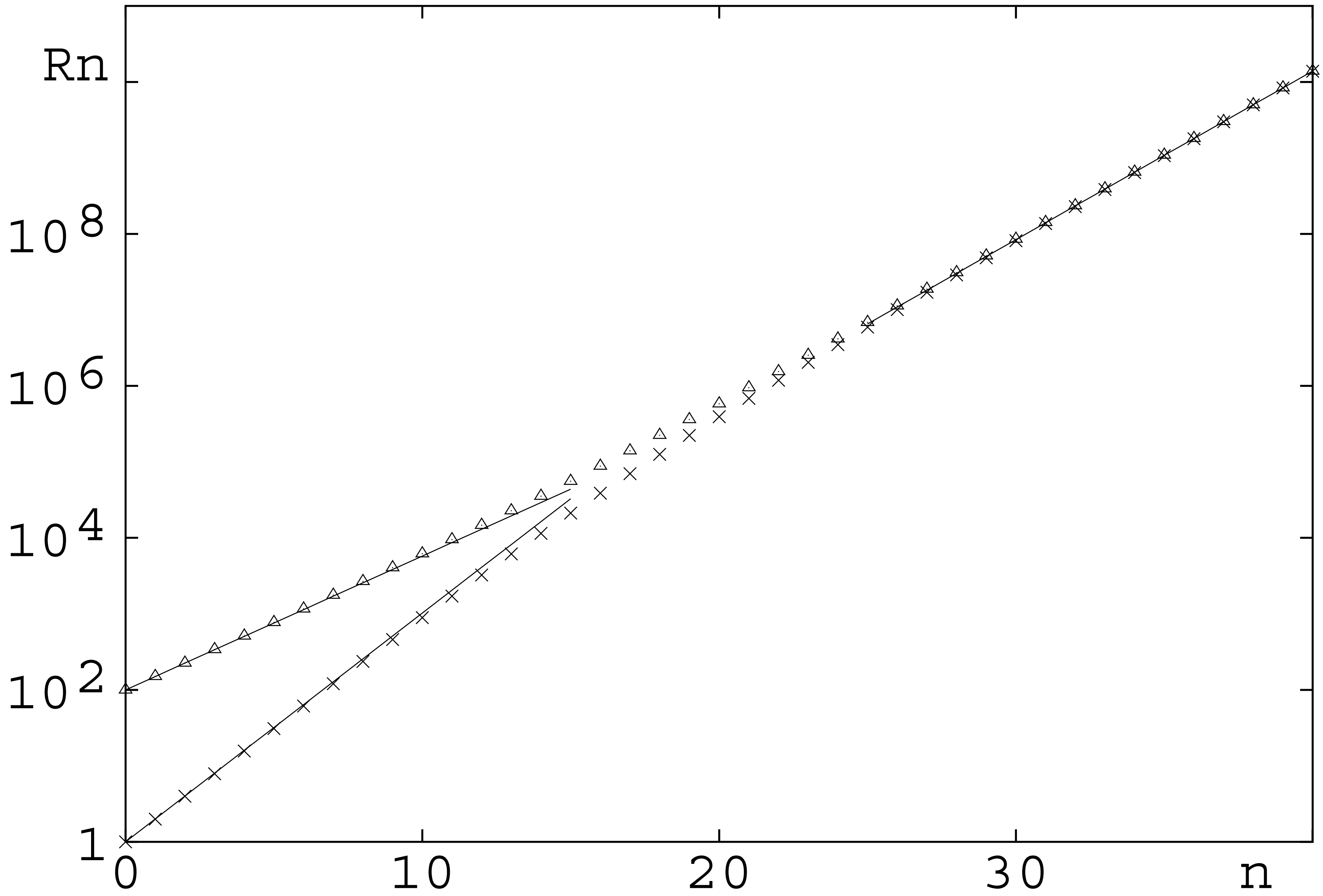

Fig. 3