Scaling limit of successive approximations for $w' = -w^2$ and its consequences on the theories of random sequential bisections and height of binary search trees

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ABSTRACT

We prove existence of scaling limits of sequences of functions defined by the recursion relation $w'_{n+1}(x) = -w_n(x)^2$. Namely, w_n approach the exact solution as $n \to \infty$ in asymptotically conformal ways, $w_n(x) \asymp q_n \bar{w}(q_n x)$, for a sequence of numbers $\{q_n\}$. We also discuss the implications of the results in terms of random sequential bisections of a rod and binary search trees.

Keywords: Scaling limit, moving singularity, successive approximation, binary search trees, random sequential bisections

1 Introduction and main results.

A solution of a linear differential equation has no singularities at regular points of the equation. On the other hand, a solution of a non-linear differential equation without singularities may have a singularity. Such a singularity may change its position among solutions with different initial conditions, hence is called a *moving singularity.* Let us consider a simplest example of such differential equations

$$\frac{dw}{dx}(x) = -w(x)^2. \tag{1}$$

While the equation (1) has no singularities, its solution

$$w(x) = \frac{1}{x - C}\,, \tag{2}$$

where C is an arbitrary constant, has a pole at $x = C$. Due to translational invariance of (1), without loss of generality we may choose $C = 0$ and consider the solution $w(x) = 1/x$ in the following. Equivalently, we may choose the boundary condition at infinity as

$$w(x) = x^{-1} + o(x^{-2}),\ x \to \infty. \tag{3}$$

Among possible ways of constructing solutions, let us consider the method of successive approximation (iteration by integration). We obtain a sequence w_n, $n = 0, 1, 2, \cdots$, of functions defined recursively, by

$$\frac{dw_{n+1}}{dx}(x) = -w_n(x)^2,\ x \geqq 0, \quad n = 0, 1, 2, \cdots, \tag{4}$$

or, equivalently,

$$w_{n+1}(x) = \int_x^\infty w_n(y)^2\,dy,\ x \geqq 0,\ n = 0, 1, 2, \cdots, \tag{5}$$

with an initial approximation

$$w_0(x) = x^{-1} + o(x^{-2}),\ x \to \infty. \tag{6}$$

Successive approximation (5) gives a sequence of functions converging to an exact solution $1/x$ of the differential equation (1).

For a solution without singularities, or in a domain where a solution is regular, the successive approximation converges uniformly to the exact solution on compact sets. In a neighborhood of a singularity of the exact solution 'uniform convergence' is obviously impossible, and we need a framework for discussing the speed of convergence there. Here we adopt renormalization group picture, and formulate our results in terms of scaling limits. To be specific, we define the scaling limit of the sequence of functions w_n, $n = 0, 1, 2, \cdots$, as a bounded function defined by

$$\bar{w}(x) = \lim_{n\to\infty} q_n^{-1} w_n(q_n^{-1} x), \tag{7}$$

for some sequence of positive numbers q_n, $n = 0, 1, 2, \cdots$, which diverges to ∞ as $n \to \infty$. Note that the scaling factor for x and the scaling factor for w_n should be equal for the present problem, because $w_n(x) \approx x^{-1}$ for very large n. Also, to have a bounded limit $\bar{w}$, q_n should diverge like $w_n(0)$, hence we shall, for simplicity, put $q_n = w_n(0)$ in the following.

Let $\mathcal{W}$ be a set of continuous functions $\bar{w}: [0,\infty) \to \mathbb{R}$, satisfying $\bar{w}(0) = 1$ and asymptotic behavior as in (6), and define $R: \mathcal{W} \to \mathcal{W}$ by

$$R(\bar{w})(x) = \frac{1}{r}\int_{x/r}^{\infty} \bar{w}(y)^2\,dy, \quad \text{where} \quad r = \int_0^{\infty} \bar{w}(y)^2\,dy. \tag{8}$$

Note that the definition of $\mathcal{W}$ implies that r in (8) is finite, hence $\bar{w} \in \mathcal{W}$ implies $R(\bar{w}) \in \mathcal{W}$. For $\bar{w}_0 \in \mathcal{W}$, define a sequence of functions $\bar{w}_n \in \mathcal{W}$, $n = 1, 2, 3, \cdots$, by

$$\bar{w}_{n+1} = R(\bar{w}_n),\ n = 0, 1, 2, \cdots. \tag{9}$$

Comparing (5) and (8) we see that if we put

$$\bar{w}_0(x) = q_0^{-1} w_0(q_0^{-1}x),\ x \geqq 0, \quad \text{with} \quad q_0 = w_0(0), \tag{10}$$

then for $n = 1, 2, \cdots$,

$$\bar{w}_n(x) = q_n^{-1} w_n(q_n^{-1}x), \quad \text{with} \quad q_n = w_n(0), \tag{11}$$

and

$$r_n = \int_0^{\infty} \bar{w}_n(y)^2\,dy \tag{12}$$

satisfying

$$r_n = \frac{q_{n+1}}{q_n}\,. \tag{13}$$

Therefore the existence of scaling limit for w_n is equivalent to the existence of $\lim_{n\to\infty} \bar{w}_n$.

The sequence $\{\bar{w}_n\}$ is uniformly bounded (in fact, $|\bar{w}_n(x)| \leqq \bar{w}_n(0) = 1$ for $n \geqq 1$), and (9) implies $\left|\frac{d\bar{w}_{n+1}}{dx}(x)\right| \leqq \frac{1}{r_n^2}\bar{w}_n^2(\frac{x}{r_n}) \leqq \frac{1}{r_n^2}$ for $n \geqq 1$, hence the Ascoli–Arzelà Theorem implies that if the sequence $\{r_n^{-2}\}$ is bounded, then $\{\bar{w}_n\}$ is relatively compact in the topology of uniform convergence on $[0,\infty)$. To go further, we will below work in a complex analytic class of functions, to ensure that the scaling limit actually exists.

To our knowledge, the problem of existence and properties of scaling limits for successive approximations to differential equations with moving singularities has not been studied or even noticed, though the recursion (4) has attracted attention [6]. A motivation of the present study is to shed first light on this potentially interesting, but somehow so far overlooked, problem.

The main results and the organization of the present paper are as follows. Denote by $\mathcal{C}$ a set of entire functions $\bar{w}: \mathbb{C} \to \mathbb{C}$, with $\bar{w}(0) = 1$, whose coefficients in the Maclaurin series have alternating signs;

$$\bar{w}(z) = \sum_{k=0}^{\infty} (-1)^k a_k z^k,\ a_k \geqq 0,\ k = 0, 1, 2, \cdots, \quad (\text{with } a_0 = 1),$$

positive on positive real axis, and asymptotic behavior in the positive real direction as in (6). In Section 2 we prove the following.

Theorem 1 *Let $\bar{w}_0 \in \mathcal{C}$ and $\bar{w}_n$, $n = 0, 1, 2, \cdots$, be a sequence defined recursively by (9). Then for each n, $\bar{w}_n$ are analytically continued to $\mathbb{C}$ and $\bar{w}_n \in \mathcal{C}$ holds. Furthermore, if the sequence (12) converges to a number greater than 1:*

$$r = \lim_{n\to\infty} r_n > 1, \tag{14}$$

then $\bar{w}_n$ converges uniformly on any compact sets in $\mathbb{C}$ to an entire function

$$\bar{w}(z) = \sum_{k=0}^{\infty} (-1)^k \alpha_k z^k \tag{15}$$

defined by

$$\alpha_0 = 1, \quad \alpha_k = \frac{1}{kr^{k+1}} \sum_{j=1}^{k} \alpha_{k-j}\alpha_{j-1}, \;\; k = 1, 2, 3, \cdots. \tag{16}$$

$\bar{w}$ also satisfies

$$\bar{w}(x) = \frac{1}{r} \int_{x/r}^{\infty} \bar{w}(y)^2 \, dy, \quad x \geqq 0, \quad \text{and} \quad \int_0^{\infty} \bar{w}(y)^2 \, dy = r. \tag{17}$$

Theorem 1 gives, through the correspondence (11), a sufficient condition for a sequence of successive approximations $\{w_n\}$ to have a scaling limit and its explicit form in terms of the Taylor coefficients α_k. In Section 3 and Section 4, we discuss detailed properties of α_k, such as asymptotics in k, and r dependences. (See, in particular, Theorem 7 and Theorem 11.)

Theorem 1 says that we have a family of possible scaling limit functions parametrized by 'the scaling factor' r of (14). Then it is of interest to know whether, given r, there exists an initial approximation w_0 satisfying (6), such that the sequence of successive approximations has a scaling limit with the scaling factor r. The following examples give affirmative answers. For $b > 2$, consider

$$w_0(x) = \frac{1}{x} (1 - e^{-x}) - \frac{1}{x^b} \gamma(b, x), \quad x \geqq 0, \tag{18}$$

where $\gamma(b,x) = \displaystyle\int_0^x y^{b-1} e^{-y} dy$ is the incomplete gamma function of first kind. Note that

$$w_0(x) = x^{-1} + O(x^{-b}), \;\; x \to \infty. \tag{19}$$

Let ρ be the unique positive solution to a transcendental equation

$$2e \log \rho = \rho < e. \tag{20}$$

To be explicit, $\rho = -2eW(\dfrac{-1}{2e}) = 1.2610704868\cdots$, where W is the Lambert W function, defined as an inverse function of $z = We^W$.

Theorem 2 *Let $2 < b < \dfrac{1}{\log \rho} = 4.31107040700\cdots$, and let w_n, $n = 0, 1, 2, \cdots$, be a sequence defined recursively by (5), with w_0 as in (18). Then the scaling limit (7) exists and satisfies (15) with (16) and (17), with*

$$r = \lim_{n \to \infty} \frac{q_{n+1}}{q_n} > 1 \tag{21}$$

given by $r = r(b)$, where

$$r(b) = \left(\frac{b}{2}\right)^{1/(b-1)}. \tag{22}$$

Note that ρ is the supremum of the function $r(b)$, and that for $2 \leqq b \leqq \frac{1}{\log \rho}$, $r(b)$ is increasing in b, with $r(2) = 1$ and $r(\frac{1}{\log \rho}) = \rho$.

Theorem 2 proves that for any r satisfying $1 < r < \rho$, there exists an initial approximation w_0 satisfying (6), such that the sequence of successive approximations has a scaling limit with r being the parameter of (14).

To prove Theorem 2, we need to prove existence of the limit (21) (and that it is given by (22)) for the specific examples given by (18). Then we can apply Theorem 1 to conclude Theorem 2. To prove that the limit (21) exists, we develop in Section 5 a monotonicity argument for propagating single layer solutions to a non-linear non-local recursion. Put

$$\begin{aligned}\Omega = \{f :\ & [0,\infty) \to [0,1] \mid \\ & \text{non-increasing, right continuous, } f(0) = 1,\ \lim_{t\to\infty} f(t) = 0\}\,.\end{aligned} \tag{23}$$

For $f \in \Omega$ define $R_1(f) :\ [0,\infty) \to [0,\infty)$ by $R_1(f)(0) = 1$ and

$$R_1(f)(t) = \frac{1}{t}\int_0^t f(s)f(t-s)\,ds,\ t > 0\,. \tag{24}$$

Obviously, $R_1(f) \in \Omega$. We prove the following in Section 5.

Theorem 3 *Let $b > 2$ and $r = r(b)$ be as in (22). Define a sequence of functions f_n, $n = 0, 1, 2, \cdots$, recursively by $f_0 = f_{b,-} \in \Omega$, where*

$$f_{b,-}(t) = \max\{1 - t^{b-1},\ 0\}, \quad t \geqq 0, \tag{25}$$

and $f_{n+1} = R_1(f_n)$, $n = 0, 1, 2, \cdots$. Then the following hold.

(i) For each $t \geqq 0$, $f_n(r^n t)$, $n = 0, 1, 2, \cdots$, is non-decreasing, hence there exists a function $\tilde{f} :\ [0,\infty) \to [0,1]$ such that

$$\lim_{n\to\infty} f_n(r^n t) = \tilde{f}(t), \quad t \geqq 0, \tag{26}$$

for which the following dichotomy, depending on b, holds: Either

(a) $\tilde{f}(t) = 1$, $t \geqq 0$, or,

(b) $\tilde{f}(t)$ is integrable:

$$Q := \int_0^\infty \tilde{f}(t)\,dt < \infty. \tag{27}$$

(ii) If in addition $b < \frac{1}{\log \rho}$, then $\tilde{f}(t) < 1$, $t > 0$, namely, the latter in the above dichotomy holds.

Theorem 3 essentially says that $\{r(b)^n\}$ is the correct scaling sequence for $2 < b < \frac{1}{\log \rho}$ (see below). It turns out that $\{r(b)^n\}$ is a wrong scaling sequence for $b > \frac{1}{\log \rho}$ (see Theorem 16 in Section 6).

The sequence $\{f_n\}$ in Theorem 3 is related to the sequence $\{w_n\}$ defined by (5) through Laplace transform. Note that (18) has an expression

$$w_0(x) = \int_0^\infty e^{-xt} f_{b,-}(t)\,dt, \quad x \geqq 0, \tag{28}$$

where $f_{b,-}$ is as in (25).

Lemma 4 *Let $f_0 \in \Omega$ and*

$$w_0(x) = \int_0^\infty e^{-xt} f_0(t)\, dt, \ \ x \geqq 0, \tag{29}$$

and let w_n, $n = 0, 1, 2, \cdots$, be a sequence defined recursively by (5). Then each w_n has an expression

$$w_n(x) = \int_0^\infty e^{-xt} f_n(t)\, dt\,, \quad x \geqq 0, \tag{30}$$

with $f_n \in \Omega$ satisfying

$$f_{n+1} = R_1(f_n), \ \ n = 0, 1, 2, \cdots. \tag{31}$$

Proof. Define $f_n \in \Omega$, $n = 0, 1, 2, \cdots$, recursively by (31), and put

$$\tilde{w}_n(x) = \int_0^\infty e^{-xt} f_n(t)\, dt\,, \ \ x \geqq 0, \ \ n = 0, 1, 2, \cdots. \tag{32}$$

If we can prove that $\tilde{w}_n = w_n$ for all n, the proof of Lemma 4 is complete. Since this holds by definition for $n = 0$, it suffices to prove that $\tilde{w}_n$ satisfies the same recursion relation (5) as w_n. But this is easy to see by calculating $\displaystyle\int_x^\infty dy \int_0^\infty du \int_0^\infty dv\, e^{-y(u+v)}\, f_n(u)\, f_n(v)$ in two ways using Fubini's Theorem. □

Proof of Theorem 2 assuming Theorem 3. Let b and $\{w_n\}$ be as in the assumptions of Theorem 2, and put $Q_n = q_n/r^n = w_n(0)/r^n$, $n = 0, 1, 2, \cdots$. Then noting the correspondence (28) and Lemma 4, we have $\displaystyle\int_0^\infty f_n(r^n t)\, dt = Q_n$. Since Theorem 3 implies that the integrand is pointwise non-decreasing in n, Q_n is non-decreasing, and the monotone convergence Theorem implies $\displaystyle\lim_{n\to\infty} Q_n = Q$. Therefore, the limit $\displaystyle\lim_{n\to\infty} \frac{q_{n+1}}{q_n} = \lim_{n\to\infty} r \frac{Q_{n+1}}{Q_n} = r$ exists, and Theorem 1, with the correspondence (10) and the explicit form (28), implies the existence of the scaling limit (7) for the choice (18), with all the consequences in Theorem 1. □

We have no analogous results to Theorem 3(ii) for $b \geqq \dfrac{1}{\log \rho}$. Also, possibility of scaling limits with $r \geqq \rho$ are not contained in Theorem 2. Concerning these points, the following extreme case turns out to be of particular interest. Put

$$w_0(x) = \frac{1}{x}(1 - \exp(-x)), \quad x \geqq 0. \tag{33}$$

Note that $w_0(x) - x^{-1}$ is exponentially small at $x \to \infty$. Note also that

$$\frac{1}{x}(1 - e^{-x}) = \int_0^\infty e^{-xt}(1 - F_0(t))\, dt \tag{34}$$

with

$$F_0(x) = \begin{cases} 0, & 0 \leqq x < 1, \\ 1, & x \geqq 1\,. \end{cases} \tag{35}$$

Theorem 5 *Let w_n, $n = 0, 1, 2, \cdots$, be a sequence defined recursively by (5), with w_0 as in (33). If a limit (21) exists with $q_n = w_n(0)$, then the scaling limit (7) exists with $r = \rho$, and satisfies (15) with (16) and (17).*

A proof of Theorem 5 in Section 6 uses a completely different approach from those for Theorem 3, and is based on a probabilistic argument for random bisection of a rod and binary search trees [3, 5, 11, 10]. See Section 6, in particular, (81), (82), and (84), for details on the implications of Theorem 5 on these subjects.

Theorem 5 (or Theorem 1) reduces the problem of convergence of a series of functions to that of a series of numbers, hence necessary numerical checks are simpler. Numerical calculations suggest that with the choice (33), q_{n+1}/q_n in fact decreases in n, hence (21) is likely to hold. Thus we conjecutre that the scaling limit exists for $\{w_n\}$ defined recursively by (5) with w_0 as in (33). The numerical results further suggests that (27) actually fails for this choice. We give details of numerical results in Appendix.

We note that a result similar to Theorem 5 holds also for (18) with $b \geqq \dfrac{1}{\log \rho}$, as a corollary to Theorem 5 and Theorem 2, using a monotonicity Lemma 12 in Section 5. See Theorem 16 in Section 6. Thus our results suggest that $r = \rho$ is the upper bound of possible r.

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2 Scaling limit.

Here we prove Theorem 1.

First we prove by induction that $\bar{w}_n \in \mathcal{C}$ for all $n = 0, 1, 2, \cdots$. Assume that $\bar{w}_n \in \mathcal{C}$ for a non-negative integer n. Positivity on $x > 0$ and $\bar{w}_{n+1}(x) = x^{-1} + o(x^{-2})$, $x \to \infty$, follows directly from the recursion (9) with (8) and $\bar{w}_n(x) = x^{-1} + o(x^{-2})$. Rewrite the recursion as

$$\bar{w}_{n+1}(x) = 1 - \frac{1}{r_n} \int_0^{x/r_n} \bar{w}_n(y)^2 \, dy, \ x \geqq 0 \,. \tag{36}$$

The integrand in the last term is entire by induction hypothesis, hence we can analytically continue $\bar{w}_{n+1}$ to the whole complex plane as an entire function, using this expression.

$\bar{w}_{n+1}(0) = 1$ also follows from (36). For a non-negative integer n, put

$$\bar{w}_n(z) = \sum_{k=0}^{\infty} (-1)^k \bar{a}_{n,k} z^k, \ z \in \mathbb{C}. \tag{37}$$

Substituting this in (36) we find $\bar{a}_{n+1,0} = 1$ and

$$\bar{a}_{n+1,k} = \frac{1}{k r_n^{k+1}} \sum_{j=1}^{k} \bar{a}_{n,k-j} \bar{a}_{n,j-1} \ (\geqq 0), \ k = 1, 2, 3, \cdots, \ n = 0, 1, 2, \cdots. \tag{38}$$

Thus by induction, $\bar{w}_n \in \mathcal{C}$ for all n.

Now we assume all the assumptions in Theorem 1, and prove the following (39) and (40):

$$\begin{array}{l}\text{There exsits } c > 0 \text{ such that the set of entire functios } \{\bar{w}_n \mid n = 0, 1, 2, \cdots\} \\ \text{is uniformly bounded on } \{z \in \mathbb{C} \mid |z| \leqq c\}.\end{array} \tag{39}$$

$$\lim_{n\to\infty} \bar{a}_{n,k} = \alpha_k, \ k = 0, 1, 2, \cdots, \tag{40}$$

where α_k is defined by (16). Ascoli–Arzelà–Montel–Vitali Theorem will then imply that $\bar{w}_n(z)$ converges uniformly to $\bar{w}(z)$ of (15) on $|z| \leqq c$. The recursion (36) with (14) then implies that $\bar{w}_n$ actually converges uniformly to $\bar{w}$ on any compact sets as $n \to \infty$, which further implies

$$\bar{w}(x) = 1 - \frac{1}{r} \int_0^{x/r} \bar{w}(y)^2 \, dy, \ x \geqq 0.$$

$\bar{w}$ is a limit of non-negative functions on $x \geqq 0$, hence is non-negative there, so that this integral equation implies square integrability of $\bar{w}$ on $[0, \infty)$, hence in particular, $\lim_{x\to\infty} \bar{w}(x) = 0$, and consequently (17) holds. Positivity follows on $x \geqq 0$, because of analyticity.

We are left with proving (39) and (40). To prove (39), put

$$a = 1 + \max\{\sup_{n\geqq 1} \frac{2}{r_n^2}, \ \max_{|z|\leqq 1} \frac{|\bar{w}_0(z) - 1|}{|z|}\}$$

which exists (is finite) by (14) and the assumption $\bar{w}_0 \in \mathcal{C}$. Then $a - \dfrac{2}{r_n^2} \geqq 1$ for all n, which further implies that there exists c, satisfying

$$0 < c < 1 \quad \text{and} \quad \frac{2}{r_n^2} + \frac{2a^2c^2}{3r_n^4} \leqq a, \quad n = 0, 1, 2, \cdots.$$

We now prove by induction that for $n = 0, 1, 2, \cdots$,

$$|\bar{w}_n(z) - 1| \leqq a|z|, \quad |z| \leqq c. \tag{41}$$

This holds for $n = 0$ by the definition of a and c. Assume that (41) holds for a non-negative integer n. Then (36) implies

$$|\bar{w}_{n+1}(z) - 1| \leqq \frac{1}{r_n} \int_0^{|z|/r_n} \bar{w}_n(y)^2 \, dy \leqq \frac{2}{r_n} \int_0^{|z|/r_n} (1 + (1 - \bar{w}_n(y))^2) \, dy \leqq a\,|z|, \quad |z| \leqq c,$$

hence (41) holds also for $n+1$, and by induction, for all n. This proves (39).

Finally, we prove (40). Let us introduce a notation which we use in the rest of this section. For $r>0$, let M_r be a map on a space of infinite sequences

$$M_r:\ a=\{a_k \mid k=0,1,2,\cdots\} \mapsto M_r(a)=\{M_r(a)_k \mid k=0,1,2,\cdots\}$$

defined by $M_r(a)_0=1$ and

$$M_r(a)_k=\frac{1}{kr^{k+1}}\sum_{j=1}^{k} a_{k-j}a_{j-1}\,,\ k=1,2,3,\cdots. \tag{42}$$

Comparing with (38), we have

$$\bar{a}_{n+1}=M_{r_n}(\bar{a}_n),\ n=0,1,2,\cdots, \tag{43}$$

where we put $\bar{a}_n=\{\bar{a}_{n,k} \mid k=0,1,2,\cdots\}$. Also, the defining equation (16) of $\alpha(r)=\alpha=\{\alpha_k \mid k=0,1,2,\cdots\}$ can be written in a concise form

$$M_r(\alpha(r))=\alpha(r). \tag{44}$$

For sequences $a=\{a_k \mid k=0,1,2,\cdots\}$ and $b=\{b_k \mid k=0,1,2,\cdots\}$ we write $a \leqq b$ if $a_k \leqq b_k$, $k=0,1,2,\cdots$. Obviously we have, for a non-negative sequence a,

$$M_r(a) \geqq M_{r'}(a),\ \text{if } 0<r \leqq r'. \tag{45}$$

Let $\gamma=\{\gamma_k \mid k=0,1,2,\cdots\}$, be a non-negative sequence satisfying $\gamma_0=1$, and for $r>0$ define $\tilde{a}(r)_n=\{\tilde{a}(r)_{n,k} \mid k=0,1,2,\cdots\}$, $n=0,1,2,\cdots$, by

$$\tilde{a}(r)_n=M_r^n(\gamma),\ n=0,1,2,\cdots. \tag{46}$$

By (45), we see that $\tilde{a}(r)_n$ is decreasing in r.

Lemma 6 *For $r>0$ and for any γ in (46), $\tilde{a}(r)_{n,k}=\alpha(r)_k$ if $n \geqq k \geqq 0$, where $\alpha(r)$ is as in (44).*

Proof. By definition, $\tilde{a}(r)_{n,0}=\alpha(r)_0=1$, $n=0,1,2,\cdots$, hence in particular the claim holds for $n=0$. Assume that the claim holds for some n and for all k satisfying $0 \leqq k \leqq n$. Then for $1 \leqq k \leqq n+1$ we have

$$\tilde{a}(r)_{n+1,k}=\frac{1}{kr^{k+1}}\sum_{j=1}^{k}\tilde{a}(r)_{n,k-j}\tilde{a}(r)_{n,j-1}=\frac{1}{kr^{k+1}}\sum_{j=1}^{k}\alpha(r)_{k-j}\alpha(r)_{j-1}=\alpha(r)_k,$$

hence the claim holds for $n+1$. □

Let us proceed with the proof of (40), and let $0<\epsilon<1$. The assumption (14) implies that there exists n_0 such that $(1-\epsilon)r \leqq r_n \leqq (1+\epsilon)r$, $n \geqq n_0$, which further implies, by induction, with (43) and (45),

$$M_{(1+\epsilon)r}^n(\bar{a}_{n_0}) \leqq \bar{a}_{n_0+n}=M_r^n(\bar{a}_{n_0}) \leqq M_{(1-\epsilon)r}^n(\bar{a}_{n_0}),\ n=0,1,2,\cdots. \tag{47}$$

Put $\gamma = \bar{a}_{n_0}$ in (46). Comparing (47) with (46) we have,

$$\tilde{a}(r(1+\epsilon))_n = M^n_{r(1+\epsilon)}(\bar{a}_{n_0}) \leqq \bar{a}_{n+n_0} \leqq M^n_{r(1-\epsilon)}(\bar{a}_{n_0}) = \tilde{a}(r(1-\epsilon))_n,\ n = 0, 1, 2, \cdots.$$

With Lemma 6 we further have,

$$\alpha(r(1+\epsilon))_k \leqq \bar{a}_{n+n_0,k} \leqq \alpha(r(1-\epsilon))_k,\ n \geqq k \geqq 0,$$

Hence

$$\alpha(r(1+\epsilon))_k \leqq \liminf_{n\to\infty} \bar{a}_{n,k} \leqq \limsup_{n\to\infty} \bar{a}_{n,k} \leqq \alpha(r(1-\epsilon))_k,\ k = 0, 1, 2, \cdots.$$

Noting that $0 < \epsilon < 1$ is arbitrary and $\alpha(r)_k$ is a polynomial in r^{-1}, we have (40).

□

3 Bounds on the Taylor coefficients of scaling limit.

Here we consider bounds and asymptotics of α_k, the Taylor coefficients (modulo signs) of the scaling limit $\bar{w}$.

Write the r dependences of the coefficients α_k in (16) explicitly, as $\alpha_k = \alpha(r)_k$. In this section, we prove the following.

Theorem 7 *For $r > 1$, there exist constants C_r, C'_r, D_r, D'_r, depending only on r, such that*

$$\alpha(r)_k \geqq \exp(-\frac{\log r}{\log 2}\,(k+1)\log(k+1) - C_r\,(k+1) + D_r),\quad k = 0, 1, 2, \ldots, \tag{48}$$

and

$$\alpha(r)_k \leqq \exp(-\frac{\log r}{\log 2}\,(k+1)\log(k+1) - C'_r\,(k+1) + D'_r),\quad k = 0, 1, 2, \cdots. \tag{49}$$

The bounds in Theorem 7 decays faster than exponentially, hence the upper bound (49) implies that the radius of convergence of (15) is ∞, which gives a direct proof that $\bar{w}$ is entire.

The upper bound (49) in Theorem 7 is easy. In fact,

$$\alpha(r)_k \leqq e^{-\log_2 r\,(k+1)\log(k+1)},\ k = 0, 1, 2, \cdots, \tag{50}$$

can be proved by induction on k, as follows.

$\alpha(r)_0 = 1$ implies (50) for $k = 0$. Assume that (50) holds for $k < k_0$. Then (16), and a change of summation variable $j = \frac{1}{2}(k_0+1) - i =: M - i$ imply

$$\begin{aligned}
\alpha_{k_0} &= \frac{1}{k_0 r^{k_0+1}} \sum_{j=1}^{k_0} \exp(-\frac{\log r}{\log 2}((k_0-j+1)\log(k_0-j+1) + j\log j) \\
&= \frac{1}{r^{k_0+1}(2M-1)} \sum_{i=-M+1}^{M-1} \exp(-\frac{\log r}{\log 2}((M+i)\log(M+i) + (M-i)\log(M-i))) \\
&\leqq \frac{1}{r^{k_0+1}} \exp(-\frac{\log r}{\log 2} \times 2M\log M) \\
&= \exp(-\frac{\log r}{\log 2}\,(k_0+1)\log(k_0+1)),
\end{aligned}$$

where we also used an elementary bound (see Proposition 8 below)

$$(M+i)\log(M+i)+(M-i)\log(M-i) \geqq 2M\log M, \quad |i|<M.$$

This proves (50) for $k=k_0$, hence (50) is proved by induction.

To go further, we first recall the following elementary facts. (Proofs will be omitted, being elementary.)

Proposition 8 *Let $a>0$, $M\geqq 1$, $\epsilon>0$, and $\Lambda>M^{-1}$. Then the following hold.*

(i) $\displaystyle\int_0^\infty e^{-Max^2}dx=\frac{1}{2}\sqrt{\frac{\pi}{Ma}}.$

(ii) $\displaystyle 0\leqq\int_\epsilon^\infty e^{-Max^2}dx\leqq\frac{1}{2Ma\epsilon}e^{-Ma\epsilon^2}\quad\left(=\int_\epsilon^\infty\frac{x}{\epsilon}e^{-Max^2}dx\right).$

(iii) $\displaystyle f_M(x)=(M+x)\log(M+x)+(M-x)\log(M-x)-2M\log M-\frac{x^2}{M}$ *is increasing in x on $0\leqq x\leqq M$, and is non-negative on $|x|\leqq M$.*

(iv) $\displaystyle\frac{1}{M}\sum_{\sqrt{M\Lambda}\leqq i\leqq M-1}e^{-ai^2/M}\leqq\int_{\sqrt{\frac{\Lambda}{M}}-\frac{1}{M}}^\infty e^{-Max^2}dx\leqq\frac{1}{2a(\sqrt{M\Lambda}-1)}e^{-a(\sqrt{\Lambda}-1/\sqrt{M})^2}$. *Here, the summation on i may be either over integers or over half odd integers, within the specified range.*

Next note the following.

Lemma 9 *Let $a>0$, and let M be a positive integer or a positive half odd integer, and put*

$$I(M,a)=\frac{1}{2M-1}\sum_{i=-M+1}^{M-1}\left(e^{-a((M+i)\log(M+i)+(M-i)\log(M-i)-2M\log M)}\sqrt{1-\frac{i^2}{M^2}}\right), \tag{51}$$

where the summation on i is over integers if M is an integer, and is over half odd integers if M is half odd. Then

$$\lim_{M\to\infty}\sqrt{M}\,I(M,a)=\frac{1}{2}\sqrt{\frac{\pi}{a}}.$$

Proof. Let $M>1$ and Λ be a real number satisfying $\dfrac{1}{M}<\Lambda<M$, and write

$$I(M,a)=K(M,a,\Lambda)+L(M,a,\Lambda), \tag{52}$$

where

$$K(M,a,\Lambda)=\frac{1}{2M-1}\sum_{\sqrt{M\Lambda}\leqq|i|\leqq M-1}e^{-a((M+i)\log(M+i)+(M-i)\log(M-i)-2M\log M)}\sqrt{1-\frac{i^2}{M^2}}, \tag{53}$$

and

$$L(M,a,\Lambda)=\frac{1}{2M-1}\sum_{|i|<\sqrt{M\Lambda}}e^{-a((M+i)\log(M+i)+(M-i)\log(M-i)-2M\log M)}\sqrt{1-\frac{i^2}{M^2}}, \tag{54}$$

where the summation on i is over integers or half odd integers, in accordance with I.

Concerning K, Proposition 8 implies

$$K(M,a,\Lambda) \leqq \frac{2}{2M-1} \sum_{\sqrt{M\Lambda} \leqq i \leqq M-1} e^{-ai^2/M} \leqq \frac{2M}{2M-1} \frac{1}{2a(\sqrt{M\Lambda}-1)} e^{-a(\sqrt{\Lambda}-1/\sqrt{M})^2},$$

hence

$$\limsup_{M\to\infty} \sqrt{M} K(M,a,\Lambda) \leqq \frac{1}{2a\sqrt{\Lambda}} e^{-a\Lambda}, \tag{55}$$

for any $\Lambda > 0$.

Next we evaluate L. Define

$$\tilde{L}(M,a,\Lambda) = \frac{1}{2M-1} \sum_{|i|<\sqrt{M\Lambda}} e^{-a((M+i)\log(M+i)+(M-i)\log(M-i)-2M\log M)},$$

and

$$L_0(M,a,\Lambda) = \frac{1}{2M-1} \sum_{|i|<\sqrt{M\Lambda}} e^{-ai^2/M}.$$

If $0 < \Lambda < M$ and $|i| < \sqrt{M\Lambda}$, we have $\left| \sqrt{1-\frac{i^2}{M^2}} - 1 \right| \leqq \frac{\Lambda}{M}$, hence

$$|L(M,a,\Lambda) - \tilde{L}(M,a,\Lambda)| \leqq \tilde{L}(M,a,\Lambda)\,\frac{\Lambda}{M},\ \ M > \Lambda > 0. \tag{56}$$

To compare $\tilde{L}$ and L_0, note that Proposition 8 implies $0 \leqq f_M(i) \leqq f_M(\sqrt{M\Lambda})$, if $|i| < \sqrt{M\Lambda}$. By elementary calculus, we further have

$$f_M(\sqrt{M\Lambda}) \leqq (2\log 2 - 1)\frac{\Lambda^2}{M}, \quad M > \Lambda > 0.$$

Therefore

$$\begin{aligned} &|\tilde{L}(M,a,\Lambda) - L_0(M,a,\Lambda)| \leqq L_0(M,a,\Lambda) \max_{|i|<\sqrt{M\Lambda}} \left| e^{-af_M(i)} - 1 \right| \\ &\leqq L_0(M,a,\Lambda) a\,(2\log 2 - 1)\frac{\Lambda^2}{M}, \quad M > \Lambda > 0. \end{aligned} \tag{57}$$

Finally, by an elementary argument of comparing summation with integration, we have, for $M > \Lambda > 0$,

$$2M \int_0^{\sqrt{\frac{\Lambda}{M}}+\frac{1}{M}} e^{-Max^2} dx - 2 \leqq (2M-1)\, L_0(M,a,\Lambda) \leqq 2M \int_0^{\sqrt{\frac{\Lambda}{M}}} e^{-Max^2} dx + 1.$$

Using Proposition 8, this eventually leads to

$$\left| \sqrt{M}\, L_0(M,a,\Lambda) - \frac{1}{2}\sqrt{\frac{\pi}{a}} \right| \leqq \frac{2M}{2M-1} \left(\frac{5}{4\sqrt{M}} + \frac{1}{2a\sqrt{\Lambda}} e^{-a\Lambda} \right). \tag{58}$$

Combining (56), (57), and (58), we arrive at

$$\limsup_{M\to\infty} \left| \sqrt{M}\, L(M,a,\Lambda) - \frac{1}{2}\sqrt{\frac{\pi}{a}} \right| \leqq \frac{1}{2a\sqrt{\Lambda}} e^{-a\Lambda}, \tag{59}$$

for any $\Lambda > 0$.

Combining (59), (55), and (52), we finally have

$$\limsup_{M\to\infty} \left| \sqrt{M}\, I(M,a) - \frac{1}{2}\sqrt{\frac{\pi}{a}} \right| \leqq \frac{1}{a\sqrt{\Lambda}} e^{-a\Lambda},$$

for any $\Lambda > 0$. The left hand side is independent of Λ, hence it must be 0.

$\square$

Theorem 10 *For $r > 1$, there exist constants C_r, C_r', D_r, D_r', depending only on r, such that*

$$\alpha(r)_k \geqq \sqrt{k+1}\exp(-\frac{\log r}{\log 2}(k+1)\log(k+1) - C_r(k+1) + D_r), \quad k = 0,1,2,\ldots, \tag{60}$$

and

$$\alpha(r)_k \leqq \sqrt{k+1}\exp(-\frac{\log r}{\log 2}(k+1)\log(k+1) - C_r'(k+1) + D_r'), \quad k = 0,1,2,\cdots. \tag{61}$$

e^{D_r} may be any number greater than $\sqrt{\frac{8\log r}{\pi\log 2}}$, and $e^{D_r'}$ any number less than $\sqrt{\frac{8\log r}{\pi\log 2}}$, (by possibly losing C_r and C_r', respectively).

Remarks. Obviously, Theorem 10 implies Theorem 7, by changing C_r, C_r', D_r, D_r', if necessary. $\diamond$

Proof of Theorem 10. Fix a positive integer k_0 arbitrarily, and put

$$e^{D_r} = \left(\inf_{k>k_0} \frac{1}{2}\sqrt{k+1}\, I(\frac{k+1}{2}, \frac{\log r}{\log 2})\right)^{-1}, \tag{62}$$

and

$$e^{D_r'} = \left(\sup_{k>k_0} \frac{1}{2}\sqrt{k+1}\, I(\frac{k+1}{2}, \frac{\log r}{\log 2})\right)^{-1}, \tag{63}$$

where I is as in (51). Note that Lemma 9 implies that, e^{D_r} and $e^{D_r'}$ are finite and positive, and that by taking k_0 sufficiently large, e^{D_r} and $e^{D_r'}$ can be made arbitrarily close to

$$\lim_{k\to\infty}\left(\frac{1}{2}\sqrt{k+1}\, I(\frac{k+1}{2}, \frac{\log r}{\log 2})\right)^{-1} = \sqrt{\frac{\pi\log 2}{8\log r}}.$$

Also, by adjusting C_r and C_r', we may always make (60) and (61) hold for finite number of k's; $k = 0,1,2,\cdots,k_0$.

Fix k_0, C_r, D_r, C_r', and D_r' as above. We have so chosen these parameters so that (60) and (61) hold for $0 \leqq k \leqq k_0$. Now assume the induction hypothesis that for some $K > k_0$,

(60) and (61) hold for $0 \leqq k < K$. Then using the recursion (16), induction hypothesis, and a change of variable $j = \dfrac{K+1}{2} - i$,

$$\begin{aligned} \alpha(r)_K &= \frac{1}{Kr^{K+1}} \sum_{j=1}^{K} \alpha_{K-j}\alpha_{j-1} \\ &\geqq \sqrt{K+1}\exp(-\frac{\log r}{\log 2}(K+1)\log(K+1) - C_r(K+1) + D_r) \\ &\quad \times e^{D_r} \times \frac{1}{2}\sqrt{K+1}\, I(\frac{K+1}{2}, \frac{\log r}{\log 2}), \end{aligned}$$

and a similar formula for the upper bound. This with Lemma 9, $K > k_0$, and (62) implies (60) for $k = K$, and (63) in place of (62) implies (61) for $k = K$.

By induction, (60) and (61) hold for all k.

□

4 Parameter dependence of the Taylor coefficients of scaling limit.

Theorem 7 and a related conjecture in Section 3 suggests that r dependences of $\alpha(r)_k$ may be of interest. Put

$$b(r)_k = \alpha(r)_k \frac{k!\, r^{k(k+3)/2}}{2^{(k-1)}}, \quad k = 0, 1, 2, \cdots. \tag{64}$$

Then b_k is a polynomial in r whose coefficients are rational.

Theorem 11 *Let j and k be non-negative integers satisfying $j < k$, Then the coefficients of r^{j-2} and lower order terms are equal for b_j and b_k. In other words, there exists B, a formal power series in r, such that the coefficients of r^{k-2} and lower order terms in b_k are equal to those of B.*

Proof. The recursion relation for b_k in (64) is, by (16),

$$b(r)_{k+1} = \frac{1}{4} \sum_{j=0}^{k} \binom{k}{j} r^{j(k-j)} b(r)_j b(r)_{k-j},, \ k = 0, 1, 2, \cdots, \quad b(r)_0 = 2. \tag{65}$$

First we have $b(r)_1 = 1$.

We see from (65) that b_k is a polynomial in r whose coefficients are rational, and that modulo $r^{2(k-2)}$, only the $j = 0, 1, k-1, k$ terms contribute in the right hand side of (65) (i.e., concerning terms of order $2(k-2)-1$ and less, only the $j = 0, 1, k-1, k$ terms count). Therefore,

$$b(r)_{k+1} \equiv b(r)_k + \frac{k}{2} r^{k-1} b(r)_{k-1} \mod r^{2(k-2)}.$$

In particular, if $b(r)_{k-1} \equiv b(r)_k \mod r^{k-2}$, then $b(r)_k \equiv b(r)_{k+1} \bmod r^{k-1}$ follows, hence by induction Theorem 11 holds.

□

Remarks. Explicitly,

$$\begin{aligned} B(r) \;=\; & 1+\frac{r}{2}+\frac{3\,r^2}{2}+2\,r^3+5\,r^4+\frac{17\,r^5}{4}+\frac{55\,r^6}{4}+\frac{51\,r^7}{4}+\frac{101\,r^8}{4}+\frac{73\,r^9}{2} \\ & +67\,r^{10}+\frac{449\,r^{11}}{8}+\frac{1161\,r^{12}}{8}+\frac{1357\,r^{13}}{8}+\frac{1069\,r^{14}}{4}+\frac{2631\,r^{15}}{8} \\ & +\frac{1099\,r^{16}}{2}+\frac{5281\,r^{17}}{8}+\frac{4859\,r^{18}}{4}+\frac{20283\,r^{19}}{16}+\cdots. \end{aligned}$$

$\diamond$

5 Monotonicity arguments.

In this section, we prove Theorem 3.

For $\ell>0$ and $f\in\Omega$ define $R_\ell(f):\ [0,\infty)\to[0,\infty)$ by $R_\ell(f)(0)=1$ and

$$R_\ell(f)(t)=\frac{1}{\ell t}\int_0^{\ell t} f(s)f(\ell t-s)\,ds,\ t>0\,. \tag{66}$$

Obviously, $R_\ell(f)\in\Omega$. We already introduced the case $\ell=1$ in (24).

Lemma 12 *(i) Let $\ell>0$ and $f\in\Omega$ and $g\in\Omega$. If $f(t)\leqq g(t)$, $t>0$, then $R_\ell(f)(t)\leqq R_\ell(g)(t)$, $t>0$.*

(ii) For $a>0$ define $T_a:\ \Omega\to\Omega$ by $T_a(f)(t)=f(at)$, $t\geqq 0$. Then

$$R_\ell=R_1\circ T_\ell, \tag{67}$$

and

$$R_\ell\circ T_a=T_a\circ R_\ell\,. \tag{68}$$

Proof. Note that

$$R_\ell(f)(t)=\int_0^1 f(\ell t\frac{1-s}{2})\,f(\ell t\frac{1+s}{2})\,ds,\quad t>0\,, \tag{69}$$

from which, with non-negativity of f and g, follows the first claim. The second claim directly follows from (66). $\square$

The following simple lemma is essential, and also shows how an apparently strange looking function (22) appears in the argument.

Lemma 13 *For $b>1$*

$$R_{r(b)}(f_{b,-})(t)\geqq f_{b,-}(t),\ t\geqq 0, \tag{70}$$

holds, where $f_{b,-}$ is as in (25).

Proof. Let $0<t\leqq 1$, and put $r=r(b)$ and $A=\{s\in(0,1]\mid f_{b,-}(rt\frac{1+s}{2})\neq 0\}$. We have $(0,1]\setminus A=\{s\in(0,1]\mid rt\frac{1+s}{2}\geqq 1\}$. Since $f_{b,-}$ is non-increasing, we have,

$$\{s\in(0,1]\mid f_{b,-}(rt\frac{1-s}{2})=0\}\subset\{s\in(0,1]\mid f_{b,-}(rt\frac{1+s}{2})=0\}\,.$$

Therefore

$$
\begin{aligned}
R_r(f_{b,-})(t) &= \int_0^1 f_{b,-}(rt\frac{1-s}{2})\, f_{b,-}(rt\frac{1+s}{2})\, ds \\
&= \int_A (1-(rt\frac{1-s}{2})^{b-1})\,(1-(rt\frac{1+s}{2})^{b-1})\, ds \\
&= \int_0^1 [1-(rt\frac{1-s}{2})^{b-1} - (rt\frac{1+s}{2})^{b-1}]\, ds \\
&\quad + \int_{(0,1]\setminus A} [(rt\frac{1+s}{2})^{b-1} - 1 + (rt\frac{1-s}{2})^{b-1}]\, ds + \int_A (\frac{1}{2}rt)^{2(b-1)}(1-s^2)^{b-1}\, ds\,.
\end{aligned}
$$

Performing the integration of the first term in the right hand side, and noting that the second and the third terms are non-negative, we have

$$
R_r(f_{b,-})(t) \geqq 1 - r(b)^{b-1}\frac{2}{b}t^{b-1} = 1 - t^{b-1} = f_{b,-}(t), \quad 0 < t \leqq 1.
$$

For $t > 1$ we have $R_r(f_{b,-})(t) \geqq 0 = f_{b,-}(t)$. □

Proof of Theorem 3. Lemma 13 and Lemma 12 implies that for each $t \geqq 0$, $R^n_{r(b)}(f_{b,-})(t) = R^n_1(f_{b,-})(r(b)^n t) = f_n(r^n t)$ is non-decreasing in n, and also $R^n_{r(b)}(f_{b,-})(t) \leqq 1$ for all $t \geqq 0$ and $n \geqq 0$. Therefore there exists a pointwise limit (26) satisfying

$$
f_{b,-}(t) \leqq \tilde{f}(t) \leqq 1, \quad t \geqq 0. \tag{71}
$$

Since this is a pointwise monotone limit, the monotone convergence Theorem and $f_{n+1} = R_r(f_n)$ with (66) imply

$$
\tilde{f}(t) = \frac{1}{rt}\int_0^{rt} \tilde{f}(s)\tilde{f}(rt-s)\, ds, \quad t > 0. \tag{72}
$$

Note also that $f_n \in \Omega$ implies that $\tilde{f}(t)$ is non-increasing in t, and that $1 < r = r(b) \leqq \rho = 1.26\cdots < 2$ for $b > 2$ implies $\frac{r}{2} > r - 1 > 0$. Using these facts in (72), we have

$$
\begin{aligned}
\tilde{f}(t) &= \frac{2}{rt}\int_0^{rt/2} \tilde{f}(s)\tilde{f}(rt-s)\, ds \\
&= \frac{2}{rt}\int_0^{(r-1)t} \tilde{f}(s)\tilde{f}(rt-s)\, ds + \frac{2}{rt}\int_{(r-1)t}^{rt/2} \tilde{f}(s)\tilde{f}(rt-s)\, ds \\
&\leqq \frac{2}{rt}\,(r-1)t\,\tilde{f}(t) + \frac{2}{rt}\int_{(r-1)t}^{rt/2} \tilde{f}(s)\tilde{f}(rt-s)\, ds,
\end{aligned}
$$

which further leads to

$$
\begin{aligned}
\tilde{f}(t) &\leqq \frac{1}{(1-\frac{r}{2})t}\int_{(r-1)t}^{rt/2} \tilde{f}(s)\tilde{f}(rt-s)\, ds \\
&\leqq \frac{1}{(1-\frac{r}{2})t}\,(\frac{1}{2}rt - (r-1)t)\,\tilde{f}((r-1)t)\,\tilde{f}(\frac{1}{2}rt) \\
&\leqq \tilde{f}((r-1)t)^2, \quad t \geqq 0.
\end{aligned} \tag{73}
$$

Put $c = \inf_{t \geqq 0} \tilde{f}(t) \in [0,1]$. Taking infimum in (73) we have $c \leqq c^2$, which, with $0 \leqq c \leqq 1$, implies $c = 0$ or $c = 1$. Assume that $\tilde{f}(t) < 1$ for some $t > 0$. Then $c = 0$, which in particular implies that there exists $t_0 > 0$ such that $\tilde{f}(t_0) \leqq e^{-1}$. For $t > t_0$, let $n = n(t) = \left[\dfrac{\log \frac{t}{t_0}}{\log \frac{1}{r-1}} \right]$, where $[x]$ denotes the largest integer not exceeding x. Note that $0 < r - 1 = r(b) - 1 \leqq \rho - 1 < 1$, hence $\delta := \dfrac{\log 2}{\log \frac{1}{r-1}} > 0$. We then have, using also monotonicity of $\tilde{f}$,

$$\tilde{f}(t) \leqq \tilde{f}((r-1)^{n(t)}t)^{2^{n(t)}} \leqq \tilde{f}(t_0)^{2^{n(t)}} \leqq e^{-2^{n(t)}} \leqq \exp(-\frac{1}{2}\left(\frac{t}{t_0}\right)^{\delta}), \quad t \geqq t_0.$$

This proves integrability (27) of $\tilde{f}$, and a proof of Theorem 3(i) is complete.

Finally, to prove Theorem 3(ii), we prepare the following lemma. Let us extend the definition of R_ℓ in (66) to non-negative right continuous functions $f: [0,\infty) \to [0,1]$ satisfying $f(0) = 1$, by $R_\ell(f)(0) = 1$ and (66). Obviously, $R_\ell(f)$ shares the properties with f, and Lemma 12 holds in this extended class of functions.

Lemma 14 *Assume that b and b' satisfy*

$$2 < b < b' \leqq \min\{\frac{1}{\log \rho},\ 2b-1\}, \tag{74}$$

and define $f_{b,b',+}: [0,\infty) \to [0,1]$ by

$$f_{b,b',+}(t) = \min\{1 - t^{b-1} + Ct^{b'-1},\ 1\}, \quad t \geqq 0,$$

where C is a constant satisfying $C \geqq \max\{1, C_1, C_2\}$, where

$$C_1 = \left(\frac{\sqrt{\pi}\,\Gamma(b) \left(\frac{r(b)}{2}\right)^{2(b-1)}}{2\,\Gamma(b+\frac{1}{2}) \left(1 - \left(\frac{r(b)}{r(b')}\right)^{b'-1}\right)} \right)^{(b'-b)/(b-1)}, \quad C_2 = \frac{b-1}{b'-1} \left(\frac{b'-b}{b'-1}\right)^{(b'-b)/(b-1)}.$$

Then

$$R_{r(b)}(f_{b,b',+})(t) \leqq f_{b,b',+}(t), \ t \geqq 0\,.$$

Remarks. Note that in the definition of C_1, $\dfrac{r(b)}{r(b')} < 1$, because $b < b' \leqq \dfrac{1}{\log \rho}$. $\diamond$

Proof. Note that $r(b') \leqq \rho = 1.26\cdots < 2$ for all $b' > 2$. Note also that $C \geqq C_2$ implies $1 - t^{b-1} + Ct^{b'-1} \geqq 0$, $t \geqq 0$, hence $f_{b,b',+}$ is non-negative on $[0,\infty)$.

Define $t_0 > 0$ by $Ct_0^{b'-b} = 1$. $C \geqq 1$ and $b' > b$ imply $t_0 \leqq 1$. Also it is easy to see that $f_{b,b',+}(t) = 1$, $t \geqq t_0$. By definition, $0 \leqq f_{b,b',+}(t) \leqq 1$, $t \geqq 0$, hence $0 \leqq R_{r(b)}(f_{b,b',+})(t) \leqq 1$, $t \geqq 0$. Therefore the statement holds for $t \geqq t_0$. In the following we assume $0 < t < t_0$.

Put, in the following, $r = r(b)$. Using $f_{b,b',+}(s) \leqq 1 - s^{b-1} + Cs^{b'-1}$, (69), and (22), we see that

$$
\begin{aligned}
R_r(f_{b,b',+})(t) &\leqq 1 - \frac{2}{b}(rt)^{b-1} + I_2(t) + I_3(t) = 1 - t^{b-1} + I_2(t) + I_3(t)\,, \quad \text{where,}\\
I_2(t) &= \left(\frac{rt}{2}\right)^{2(b-1)} \int_0^1 (1-s^2)^{b-1}\,ds + \frac{2C}{b'}(rt)^{b'-1}\\
&= Ct^{b'-1}\left(\frac{1}{C}\left(\frac{r}{2}\right)^{2(b-1)} \int_0^1 (1-s^2)^{b-1}\,ds\, t^{2b-b'-1} + \left(\frac{r}{r(b')}\right)^{b'-1}\right),\\
I_3(t) &= -C\left(\frac{rt}{2}\right)^{b+b'-2} \int_0^1 (1+s)^{b-1}(1-s)^{b'-1}\,ds\\
&\quad - C\left(\frac{rt}{2}\right)^{b+b'-2} \int_0^1 (1-s)^{b-1}(1+s)^{b'-1}\left(1 - C\left(\frac{rt}{2}\right)^{b'-b}(1-s)^{b'-b}\right) ds.
\end{aligned}
$$

Using $t < t_0 = C^{-1/(b'-b)}$ with $2b - b' - 1 \geqq 0$, and $C \geqq C_1$ with $b' > b > 1$, we find $I_2(t) \leqq Ct^{b'-1}$. Using $Ct_0^{b'-b} = 1$ and $r < r(b') < 2$, we have $C\left(\frac{rt}{2}\right)^{b'-b} \leqq 1$, hence $I_3(t) \leqq 0$. We therefore have $R_r(f_{b,b',+})(t) \leqq 1 - t^{b-1} + Ct^{b'-1} \leqq f_{b,b',+}(t)$. $\square$

Let us return to a proof of Theorem 3(ii), and let $2 < b < \dfrac{1}{\log \rho}$ and $r = r(b)$. Note that for such a b, there exists b' which satisfies (74). Lemma 12(i) and Lemma 14 imply, for $f_0 = f_{b,-}$,

$$f_n(r^n t) = R_r^n(f_{b,-})(t) \leqq R_r^n(f_{b,b',+})(t) \leqq f_{b,b',+}(t),\ t \geqq 0.$$

Taking a limit $n \to \infty$, we then have $\tilde{f}(t) \leqq f_{b,b',+}(t)$, $t \geqq 0$. Therefore $\tilde{f}(t) < 1$ for all small positive t, and monotonicity of $\tilde{f}(t)$ in t further implies $\tilde{f}(t) < 1$, $t > 0$. This completes a proof of Theorem 3.

$\square$

6 Random sequential bisections of a rod.

Here we prove Theorem 5, and discuss its implications in terms of random sequential bisections of a rod and in binary search trees. We also prove Theorem 16 as a corollary to Theorem 5, Theorem 2, and Lemma 12.

That w_0 in (33) is in $\mathcal{C}$ is obvious, if we note $q_0 = w_0(0) = 1$ and

$$\frac{1}{z}(1 - \exp(-z)) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(k+1)!} z^k.$$

If we note the correspondences (11) and (13), Theorem 1 implies all the conclusions of Theorem 5 except the claim $r = \rho$. Now, the value r of the limit (21), which we assumed to exist in Theorem 5, is equal to that of a (weaker) limit $\lim\limits_{n\to\infty} q_n^{1/n}$. This weaker limit can be derived by applying a theory of random sequential bisection of a rod, as follows.

Let F_0 be as in (35), and define a sequence of functions $F_n: [0,\infty) \to [0,1]$, $n = 0,1,2,\cdots$, recursively, by

$$1 - F_{n+1}(t) = \frac{1}{t}\int_0^t (1-F_n(s))(1-F_n(t-s))\,ds. \tag{75}$$

It is noted in eq. (5.1) of [11] that $1-F_n(x)$ thus defined is the probability that at the nth stage of random sequential bisections of a rod of length x, all the pieces have length shorter than 1. Namely, one starts with a rod of length x and breaks it into two pieces randomly with uniform distribution. Then one breaks each of the resulting two pieces randomly and independently, and contiue the procedure recursively, and see, after n steps, whether all the 2^n pieces are shorter than 1. Alternatively, one could start from a rod of length 1, and denote by X_n the length of the longest piece among 2^n pieces at nth stage, starting from $X_0 = 1$. Then

$$1 - F_n(1/t) = \mathrm{Prob}[\, X_n < t\,]. \tag{76}$$

Lemma 4 with (75) implies

$$w_n(x) = \int_0^\infty e^{-xt}(1-F_n(t))\,dt,\ x \geqq 0,\ n = 0,1,2,\cdots, \tag{77}$$

and

$$q_n = w_n(0) = \mathrm{E}[\,\frac{1}{X_n}\,], \tag{78}$$

where we denote the expectations by $\mathrm{E}[\,\cdot\,]$.

First note that, the existence of a limit $\lim_{n\to\infty} q_n^{1/n}$ can be proved without the assumption (21) in Theorem 5 of the existence of a stronger limit. In fact, consider the longest piece $\tilde{X}_{n,m}$ at $n+m$th stage among descendants from the longest piece X_n at nth stage. Clearly $\tilde{X}_{n,m} \leqq X_{n+m}$. Note also that $\tilde{X}_{n,m}/X_n$ and X_n are independent and the former is equal in distribution to X_m. Therefore

$$q_{n+m} = \mathrm{E}[\,\frac{1}{X_{n+m}}\,] \leqq \mathrm{E}[\,\frac{1}{\tilde{X}_{n+m}}\,] = \mathrm{E}[\,\frac{X_n}{\tilde{X}_{n,m}}\,]\mathrm{E}[\,\frac{1}{X_n}\,] = \mathrm{E}[\,\frac{1}{X_m}\,]\mathrm{E}[\,\frac{1}{X_n}\,] = q_m q_n\,.$$

Hence

$$q_{n+m} \leqq q_n q_m\,,\ n,m = 1,2,3,\cdots. \tag{79}$$

Using standard arguments on subadditivity, we deduce that the limit $\lim_{n\to\infty} {q_n}^{1/n} = \inf_{n\geqq 1} {q_n}^{1/n}$ exists.

It remains to prove $\lim_{n\to\infty} {q_n}^{1/n} = \rho$ for a proof of Theorem 5. The bound $\lim_{n\to\infty} {q_n}^{1/n} \geqq \rho$ is not difficult. We may, for example, apply Lemma 15 below with (35) for $1 - f(x)$ and with $b = \dfrac{1}{\log\rho}$ (i.e., $r(b) = \rho$) to conclude $\lim_{n\to\infty} q_n^{1/n} \geqq \rho$. Alternatively, we may use

$$\lim_{n\to\infty} X_n^{-1/n} = \rho,\quad a.s., \tag{80}$$

which is a result [2, Cor. to Thm. 2] applied to the problem of random sequential bisection along the lines of [3]. Using (78), Hölder's inequality, Fatou's Lemma, and (80) in turn, we have

$$\lim_{n\to\infty} q_n^{1/n} = \lim_{n\to\infty} \mathrm{E}[\,\frac{1}{X_n}\,]^{1/n} \geqq \liminf_{n\to\infty} \mathrm{E}[\,X_n^{-1/n}\,] \geqq \mathrm{E}[\,\lim_{n\to\infty} X_n^{-1/n}\,] = \rho.$$

Finally we prove the bound in the other direction, $\lim_{n\to\infty} {q_n}^{1/n} \leqq \rho$. Almost sure convergence implies convergence in probability, hence (80) implies

$$\text{Prob}[\lim_{n\to\infty} \frac{1}{X_n} \geqq (\rho+\epsilon)^n\,] = 0$$

for any $\epsilon > 0$. This with (76) and the dominated convergence Theorem implies, for each $x > 0$,

$$\lim_{n\to\infty} \int_0^\infty (1 - F_n((\rho+\epsilon)^n t))\, e^{-xt}\, dt = \int_0^\infty \lim_{n\to\infty} \text{Prob}[\,\frac{1}{X_n} > (\rho+\epsilon)^n t\,]\, e^{-xt}\, dt = 0\,.$$

Assume that the claim is wrong, so that $\lim_{n\to\infty} q_n^{1/n} > \rho + \epsilon$ for some $\epsilon > 0$. Then for large enough n we have $q_n > (\rho+\epsilon)^n$. Note that (76) implies that $F_n(s)$ is increasing in s. Therefore, using also (77) and the existence of scaling limit $\bar{w}$ (which is already proved under the assumptions of Theorem 5, as noted at the beginning of this section),

$$\lim_{n\to\infty} \int_0^\infty (1 - F_n((\rho+\epsilon)^n t))\, e^{-xt}\, dt \geqq \lim_{n\to\infty} \int_0^\infty (1 - F_n(q_n t))\, e^{-xt}\, dt = \bar{w}(x).$$

Hence $0 \geqq \bar{w}(x)$, $x > 0$, resulting in a contradiction. Thus the claim $\lim_{n\to\infty} q_n^{1/n} \leqq \rho$ is proved, and $r = \lim_{n\to\infty} q_n^{1/n} = \rho$ holds.

This completes a proof of Theorem 5.

Remarks. (i) Using the explicit formula (16) in Theorem 1, we can compare our results with known probability distributions, for example, the extreme value distributions in [9], which are possible scaling limits $n \to \infty$ of maximum of n i.i.d. random variables. We find that for $r > 1$ our scaling limits are not in [9]. Since the random sequential bisections produce pieces of non-independent lengths this looks natural, but we are further saying that in the $n \to \infty$ limit, the dependences remains, and our problem is non-trivial also from this aspect.

(ii) As we have seen, $\lim_{n\to\infty} q_n^{1/n} = \rho$ is proved by applying existing results in [2, 3]. It may seem puzzling why the existence of $\lim_{n\to\infty} q_{n+1}/q_n$ is nevertheless unproved. A direct explanation is that the arguments in [2, 3] is based on large deviation principles or the law of large numbers type arguments [1, 2, 7, 8], which are strong enough to control $q_n^{1/n}$, but unfortunately cannot control q_n strongly enough to prove the existence of $\lim_{n\to\infty} q_{n+1}/q_n$. In fact, the results of [2] imply (as used in [3]) that instead of random sequential bisections, if we, for each piece at each stage of construction of rod pieces, generated 2 independent random variables each with uniform distribution on $[0, 1]$ and creating 2 pieces whose lengths are those given by multiplying the length of the parent piece with the so generated random variables, and repeat the procedure, then $\tilde{q}_n$, the expectation value of the inverse maximum length among the 2^n pieces at n-th stage, also satisfies $\lim_{n\to\infty} \tilde{q}_n^{1/n} = \rho$. Thus the scaling factor ρ does not contain the information of anti-correlation of the lengths of the 2 pieces obtained by breaking a rod piece. The recursion relation is different from (75) for this new problem, hence the scaling limits (if they exist) are different. Hence we may say that the existence problem of $\lim_{n\to\infty} q_{n+1}/q_n$ contains the information of correlation of bisected pieces (through Theorem 1), and are therefore deeper than an already profound theory in [2].

(iii) The case $1 < r < \rho$ (see Theorem 2 and Section 5) corresponds, in terms of random sequential bisections, to the case where the initial rod is given a random length (with possibility of indefinitely large length), which does not usually seem to appear in practical situations. Of course, from differential equation point of view, problem of scaling factor different from ρ for different initial functions makes sense.

$\diamond$

We will turn to some remarks on the implication of Theorem 5 on random sequential bisections of a rod and on binary search trees. Theorem 5 in particular implies that the distribution of $\dfrac{1}{q_n X_n}$ converges weakly to a distribution (*scaling limit*) whose generating function is given by

$$\lim_{n\to\infty} \mathrm{E}[\, e^{-z/(q_n X_n)}\,] = 1 - z\bar{w}(z), \quad z \in \mathbb{C}, \tag{81}$$

and that the moments satisfy

$$\lim_{n\to\infty} \frac{\mathrm{E}[\, X_n^{-k}\,]}{\mathrm{E}[\, X_n^{-1}\,]^k} = \alpha_{k-1} k!, \quad k = 2, 3, 4, \cdots. \tag{82}$$

with α_k given by (16) (assuming a numerically supported conjecture that $\lim\limits_{n\to\infty} q_{n+1}/q_n$ exists).

We also note that the maximal length X_n of random sequential bisections of a rod is closely related to the height H_N of binary search trees with data size N. In fact, it is noted in [3] that

$$\mathrm{Prob}[\, X_n \geqq \frac{1+n}{N}\,] \leqq \mathrm{Prob}[\, H_N \geqq n\,] \leqq \mathrm{Prob}[\, X_n \geqq \frac{1}{N}\,], \quad n, N \in \mathbb{N}.$$

This is not completely sufficient to proceed further, but if we could further assume, for example, that

$$\mathrm{Prob}[\, H_N \leqq n\,] \sim \mathrm{Prob}[\, X_n \leqq \frac{n^\beta}{N}\,] \tag{83}$$

asymptotically (in some good sense) for a $0 \leqq \beta \leqq 1$, then (82) would give, by noting $\mathrm{E}[\, X^{-k}\,] = k \displaystyle\int_0^\infty x^{-k-1} \mathrm{Prob}[\, X^{-1} \leqq x\,] dx$,

$$\lim_{n\to\infty} \frac{\displaystyle\sum_{N=1}^\infty N^k \mathrm{Prob}[\, H_N \leqq n\,]}{(\displaystyle\sum_{N=1}^\infty \mathrm{Prob}[\, H_N \leqq n\,])^{k+1}} = \alpha_k k!. \tag{84}$$

(Note that this formula is independent of β, hence (84) might be correct irrespective of the validity of the extra assumption (83).) Possibility of such 'sum rules' seems not to have been considered.

To conclude this section, we note a result for (18) with $b \geqq \dfrac{1}{\log \rho}$ (Theorem 16).

Lemma 15 *For $f \in \Omega$, let*

$$q_n[f] = \int_0^\infty R_1^n(f)(s)\, ds, \tag{85}$$

where R_1^n is the n-fold iteration of R_1. (Similar notations for the iterations will be used hereafter.)

(i) If, for $f \in \Omega$, there exist constants $b > 1$ and $C > 0$ such that $f(t) \geqq 1 - Ct^{b-1}$, $t \geqq 0$, then $\liminf_{n\to\infty} q_n[f]^{1/n} \geqq r(b)$.

(ii) If, for $f \in \Omega$, there exists a constant $t_0 > 0$ such that $f(t) = 0$, $t \geqq t_0$, then $\limsup_{n\to\infty} q_n[f]^{1/n} \leqq \rho$.

(iii) If, for $f \in \Omega$, there exist constants $b \geqq \dfrac{1}{\log\rho}$, $C > 0$, and $t_0 > 0$, such that $f(t) \geqq 1 - Ct^{b-1}$, $t \geqq 0$, and $f(t) = 0$, $t \geqq t_0$, then $\lim_{n\to\infty} q_n[f]^{1/n} = \rho$.

Proof. (i) Put $\tilde f(t) = f(C^{-1/(b-1)}t)$, $t \geqq 0$. Then $\tilde f(t) \geqq f_{b,-}(t)$, $t \geqq 0$. Lemma 12 and Lemma 13 therefore imply

$$R_1^n(f)(C^{-1/(b-1)}r(b)^n t) = R_{r(b)}^n(\tilde f)(t) \geqq R_{r(b)}^n(f_{b,-})(t) \geqq f_{b,-}(t), \quad t \geqq 0,$$

or

$$R_1^n(f)(t) \geqq f_{b,-}(C^{1/(b-1)}r(b)^{-n}t), \quad t \geqq 0,$$

and consequently $q_n[f] \geqq C^{-1/(b-1)}r(b)^n(1 - \dfrac{1}{b})$.

(ii) Note that F_n defined by (75) and (35) satisfies $1 - F_n(t) = R_1^n(1 - F_0)(t)$, $t \geqq 0$, where F_0 is defined by (35). On the other hand, the assumptions imply

$$1 - F_0(t) \geqq f(t_0 t) = T_{t_0}(f)(t), \quad t \geqq 0.$$

Therefore Lemma 13 implies

$$q_n = \int_0^\infty (1 - F_n(t))\,dt \geqq \int_0^\infty R_1^n(f)(t_0 t)\,dt = \frac{1}{t_0}\,q_n[f],$$

where q_n is as in (78). As proved before, $\rho = \lim_{n\to\infty} q_n^{1/n}$ holds, hence the claim follows.

(iii) The assumptions imply that there exists a constant $C' > 0$ such that $f(t) \geqq 1 - C' t^{1/\log\rho}$, $t \geqq 0$ (because $t^b \leqq t^{1/\log\rho}$, $0 \leqq t \leqq 1$). Then the first claim implies $\liminf_{n\to\infty} q_n[f]^{1/n} \geqq r(\dfrac{1}{\log\rho}) = \rho$. The assumptions also imply that the second claim can also be applied to have $\limsup_{n\to\infty} q_n[f]^{1/n} \leqq \rho$. Hence the limit exists and the last claim follows. □

Theorem 16 *Let $b \geqq \dfrac{1}{\log\rho}$, and let w_n, $n = 0, 1, 2, \cdots$, be a sequence defined recursively by (5), with w_0 as in (18). If (21) holds with $q_n = w_n(0)$, then the scaling limit (7) exists with $r = \rho$, and satisfies (15) with (16) and (17).*

Proof. Note that w_0 in (18) has an expression (28). In particular, the corresponding f_0 in (30) is $f_0 = f_{b,-}$, which satisfies $f_0(t) = 0$ for $t \geqq 1$. Then (29) implies

$$w_0(x) = \sum_{k=0}^\infty \frac{(-x)^k}{k!} \int_0^1 t^k f_0(t)\,dt, \tag{86}$$

which has an infinite radius of convergence, hence can be continued analytically to an entire function $w_0 : \mathbb{C} \to \mathbb{C}$. Also (86) and (29) imply that $\bar{w}_0$ defined by (10) is in the class $\mathcal{C}$ defined just before Theorem 1. Therefore Theorem 1 with the correspondences (11) and (13) implies that the scaling limit $\bar{w}$ exists and satisfies (15) with (16) and (17).

The only remaining claim to prove is $r = \rho$, where $r = \lim_{n\to\infty} \dfrac{q_{n+1}}{q_n}$. Note that the limit is equal to a weaker limit $\lim_{n\to\infty} q_n^{1/n}$. Lemma 4 implies that w_n has an expression (77) with $f_n \in \Omega$ satisfying $f_n = R_1^n(f_0)$. Lemma 15 implies, by noting

$$q_n = w_n(0) = \int_0^\infty f_n(t)\,dt = q_n[f_n],$$

that $r = \rho$ if $b \geqq \dfrac{1}{\log \rho}$, which completes a proof of Theorem 16. $\square$

A Appendix.

Concerning the assumption (21) in Theorem 1, we give numerical results of the ratio q_{n+1}/q_n for the case (33), in Fig. 1.

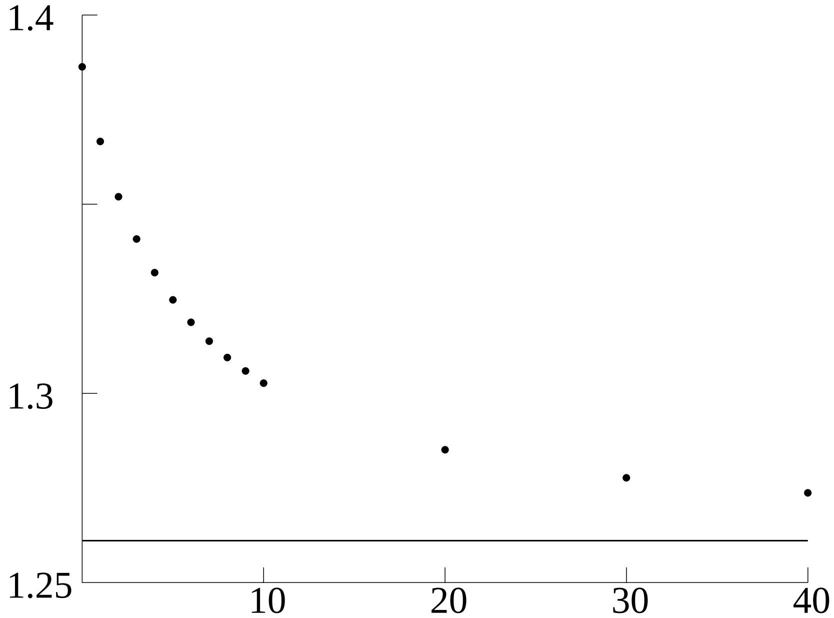

Figure 1: Plot of numerical results for q_{n+1}/q_n vs n. The number of sample points $N = 3200$. The differences between $N = 1600$ and $N = 3200$ results are within the size of plots in the figure.

$q_1/q_0 = 2\log 2$ can be obtained explicitly, but for $n \geqq 1$, we have to perform numerical calculations. Numerical values are obtained by discretizing $1 - F_n$ (i.e., represent the function by its values at a finite number, say N, of points), and performing numerical integration (i.e., approximating by a discrete sum of N terms) of (31), starting from (35). To be more explicit, we represented F_n by $F_n(x_{n,i})$, $i = 0, 1, 2, \cdots, N-1$, where $x_{n,i} = 2^{ni/N}$.

The results suggest that q_{n+1}/q_n is decreasing in n. In fact, $q_2/q_1 \leqq q_1/q_0$, or equivalently, $q_2 \leqq q_1^2$ does hold, by substituting $n = m = 1$ in (79). However, $q_3/q_2 \leqq q_2/q_1$, or $q_3 q_1 \leqq q_2^2$, already seems hard to prove. Our numerical results are also consistent with $\lim_{n\to\infty} q_{n+1}/q_n = \rho = 1.261\cdots$.

Concerning the integrability of Q in (27), our numerical data for $Q_n = q_n/\rho^n$ given in Fig. 2 suggests that Q_n is increasing in positive power order and $Q = \lim_{n\to\infty} Q_n = \infty$ (see the proof of Theorem 3 in Section 5), hence ρ^n is not a correct scaling factor for the present case, and $q_n = w_n(0)$ should be used, as discussed in Section 1.

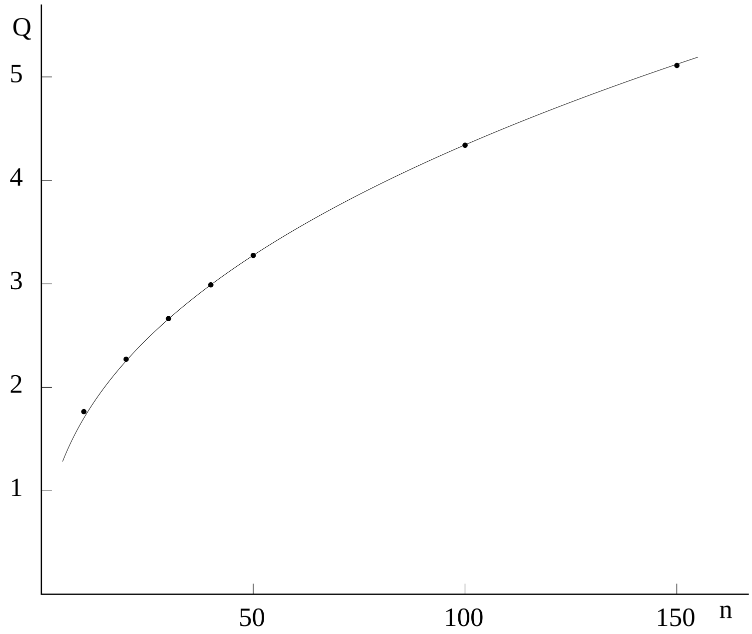

Figure 2: Plot of $Q_n = q_n/\rho^n$ vs n. The curve is a fit to the data: $Q_n = 0.666\, n^{0.407}$.

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