A Note on the Market Beta and the Consumption Beta

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This note clarifies the difference between two explanations of rates of return on risky assets, market beta and consumption beta theories, following Mankiw and Shapiro(1986). The framework of analysis is due to Sharpe (1964) and Mossin (1966), and also to Samuelson (1969). This author's view is that there is no fundamental difference between the two. Consumption beta theory gives us an added insight into the problem, however. Blanchard and Fischer (1989) is an easy introduction to the problem.

1 Market Beta

1.1 Definition of Market Beta

Consider a capital asset market where there are n risky assets and one riskless asset. The total return to the ith asset is y_i with mean μ_i and variance $\sigma_{ii}, i = 1, \ldots, n$. Further, the covariance of y_i and y_j is $\sigma_{ij}, i, j = 1, \ldots, n$. When the total market value of the ith asset is v_i, its rate of return ρ_i is defined by

$$1 + \rho_i = \frac{y_i}{v_i} \tag{1}$$

Denote the total return and the total market value of the riskless asset by y_0 and v_0, respectively. Then the riskless rate of return r is defined by

$$1 + r = \frac{y_0}{v_0} \tag{2}$$

Let $y = \sum_{i=0}^{n} y_i$ and $v = \sum_{i=0}^{n} v_i$ respectively be the total return and the total market value of all the assets, and define the market rate of return ρ_M by

$$1 + \rho_M = \frac{y}{v} \tag{3}$$

The market beta for the ith asset, β_{Mi}, i=1, ..., n, is defined by

$$\beta_{Mi} = \frac{\mathrm{cov}(\rho_i, \rho_M)}{\mathrm{var}(\rho_M)} \tag{4}$$

1.2 Linearity

1.2.1 Linearity Hypothesis

Suppose there is a linear relationship

$$E(\rho_i) = a + b\beta_{Mi}$$

between the expected rate of return and the market beta. Extend the hypothesis to include the riskless asset, for which the market beta, β_{M0}, is zero, and the market portfolio, for which the market beta, β_{MM}, is unity. Then,

$$r = a$$
$$E(\rho_M) = a + b$$

Hence

$$E(\rho_i) = r + [E(\rho_M) - r]\beta_{Mi} \tag{5}$$

1.2.2 Market Line and Market Beta

The market line of Sharpe and Mossin implies the supposed linearity, (5). To show this, express variances and covariances of rates of return using σ_{ij} and v_i.

$$\begin{aligned}
\mathrm{cov}(\rho_i, \rho_M) &= E\left[\left(\frac{y_i}{v_i} - \frac{\mu_i}{v_i}\right)\left(\frac{y}{v} - \frac{\mu}{v}\right)\right] \\
&= \frac{1}{v_i v} E[(y_i - \mu_i)\sum_{j=0}^{n}(y_j - \mu_j)] \\
&= \frac{1}{v_i v} \sum_{j=1}^{n} E[(y_i - \mu_i)(y_j - \mu_j)] \\
&= \frac{\sum_j \sigma_{ij}}{v_i v}
\end{aligned}$$

$$\begin{aligned}
\mathrm{var}(\rho_M) &= E\left[\left(\frac{y}{v} - \frac{\mu}{v}\right)^2\right] \\
&= \frac{1}{v^2} E\left[\sum_i (y_i - \mu_i)^2\right] \\
&= \frac{\sum_i \sum_j \sigma_{ij}}{v^2}
\end{aligned}$$

where μ is the mean of y. Therefore

$$\frac{\mathrm{cov}(\rho_i, \rho_M)}{\mathrm{var}(\rho_M)} = \frac{\sum_j \sigma_{ij}}{\sum_i \sum_j \sigma_{ij}} \frac{v}{v_i} \tag{6}$$

On the other hand, as Mossin (1966) showed, the equation for the market line is

$$\frac{z_{k1}}{w_k} = (1 + r) + \lambda_k \frac{\sqrt{z_{k2}}}{w_k} \tag{7}$$

for the kth household, where

$$\lambda_k = \frac{[\mu_i - (1 + r)v_i]\sqrt{\sum_i \sum_j \sigma_{ij}}}{\sum_j \sigma_{ij}}$$

z_{k1} and z_{k2} are mean and variance of the return to the kth household's portfolio, w_k being its wealth. Notice that the value of λ is the same for all k.

Transforming (7), we obtain the linear relationship between the expected rate of return and the market beta. Firtst, let θ_{ki} be the kth household's share of the ith risky asset. Then, as every household has the same share of all the risky assets in equilibrium, we have

$$z_{k2} = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \theta_k^2$$

where θ_k is the common value of $\theta_{ki}, i = 1, 2, \cdots, n$. Therefore

$$\frac{z_{k1}}{w_k} = (1+r) + \frac{\theta_k[(\mu_i/w_k) - (1+r)(v_i/w_k)]\left(\sum_i \sum_j \sigma_{ij}\right)}{\sum_j \sigma_{ij}}$$

and

$$z_{k1} = (1+r)w_k + \frac{\theta_k v_i [E(\rho_i) - r]\left(\sum_i \sum_j \sigma_{ij}\right)}{\sum_j \sigma_{ij}}$$

Sum over k to obtain

$$\mu = (1+r)v + v_i[E(\rho_i) - r]\frac{\sum_i \sum_j \sigma_{ij}}{\sum_j \sigma_{ij}}$$

and finally

$$E(\rho_i) = r + [E(\rho_M) - r]\frac{\sum_j \sigma_{ij}}{\sum_i \sum_j \sigma_{ij}} \frac{v}{v_i} \tag{8}$$

which is equivalent to (5) because of (6) and the definition of market beta, (4).

2 Consumption Beta

2.1 Definition of Consumption Beta

Let c_t be per capita consumption at time t, and define consumption growth rate γ by

$$1 + \gamma = \frac{c_{t+1}}{c_t} \tag{9}$$

Then, the consumption beta is defined as

$$\beta_{Ci} = \frac{\text{cov}(\rho_i, \gamma)}{\text{cov}(\rho_M, \gamma)} \tag{10}$$

The estimated β_{Ci} is the instrumental variable estimate of β_{Mi}. The former, therefore, is a consistant estimator of β_{Mi}, provided that γ is uncorrelated with ρ_i and ρ_M.

2.2 Linearity

2.2.1 Linearity Hypothesis

Now suppose there is a linear relationship

$$E(\rho_i) = a + b\beta_{Ci}$$

between the expected rate of return and the consumption beta. For the same reason as for (5), we have

$$r = a$$
$$E(\rho_M) = a + b$$

and

$$E(\rho_i) = r + [E(\rho_M) - r]\beta_{Ci} \tag{11}$$

2.2.2 Liftetime Portfolio Selection and Consumption Beta

Intertemporal Marginal Rate of Substitution and the Rate of Return The optimium conditions for the lifetime portfolio selection problem garantee the linearity between the expected rate of return and the consumption beta. Remember that optimum consumption and portfolio must satisfy

$$E\left[(\rho_i - r)u_c(c_{t+1})\right] = 0 \tag{12}$$
$$\frac{u_c(c_t)}{\alpha E\left[u_c(c_{t+1})\right]} = 1 + r \tag{13}$$

where, u is the utility function, and α, the psychological discount factor of the representative household. This set of optimum conditions implies that

$$E\left[\frac{(1+\rho_i)\alpha u_c(c_{t+1})}{u_c(c_t)}\right] = E\left[\frac{(1+r)\alpha u_c(c_{t+1})}{u_c(c_t)}\right] = 1, \quad \text{for all } i.$$

Applying the well known formula of covariance

$$\operatorname{cov}(x, y) = E(xy) - E(x)E(y)$$

to the covariance between the rate of return and the intertemporal marginal rate of substitution, we have

$$\operatorname{cov}\left[\rho_i,\ \frac{\alpha u_c(c_{t+1})}{u_c(c_t)}\right] = E\left[\rho_i\ \frac{\alpha u_c(c_{t+1})}{u_c(c_t)}\right] - E(\rho_i)\ E\left[\frac{\alpha u_c(c_{t+1})}{u_c(c_t)}\right]$$

Therefore,

$$\operatorname{cov}\left[\rho_i,\ \frac{\alpha u_c(c_{t+1})}{u_c(c_t)}\right] = E\left[(\rho_i - r)\frac{\alpha u_c(c_{t+1})}{u_c(c_t)}\right] - \left[E(\rho_i) - r\right] E\left[\frac{\alpha u_c(c_{t+1})}{u_c(c_t)}\right]$$

and using the first of the optimum condition, (12),

$$\operatorname{cov}\left[\rho_i,\ \frac{\alpha u_c(c_{t+1})}{u_c(c_t)}\right] = -\left[E(\rho_i) - r\right] E\left[\frac{\alpha u_c(c_{t+1})}{u_c(c_t)}\right]$$

Solving for $E(\rho_i)$, we have

$$E(\rho_i) = r - \frac{\operatorname{cov}\left[\rho_i,\ \dfrac{\alpha u_c(c_{t+1})}{u_c(c_t)}\right]}{E\left[\dfrac{\alpha u_c(c_{t+1})}{u_c(c_t)}\right]}$$

Rewrite, denoting the intertemporal marginal rate of substitution by s, to obtain

$$E(\rho_i) = r - \frac{\mathrm{cov}(\rho_i,\ s)}{E(s)} \tag{14}$$

Since ρ_M is a weighted average of $\rho_i, i = 1, \ldots, n$, we have a similar relationship for ρ_M:

$$E(\rho_M) = r - \frac{\mathrm{cov}(\rho_M,\ s)}{E(s)} \tag{15}$$

From (14) and (15)

$$E(\rho_i) = r + [E(\rho_M) - r]\frac{\mathrm{cov}(\rho_i, s)}{\mathrm{cov}(\rho_M, s)} \tag{16}$$

Approximation Now, if the degree of relative risk aversion is constant equal to η, and utility funciton is of the form

$$u(c) = \frac{c^{1-\eta}}{1-\eta} \tag{17}$$

and, therefore,

$$\frac{u_c(c_{t+1})}{u_c(c_t)} = \left(\frac{c_{t+1}}{c_t}\right)^{-\eta}$$

the intertemporal marginal rate of substitution is approximated by $\alpha(1 - \eta\gamma)$, and therefore,

$$\mathrm{cov}(\rho_i, s) \approx -\alpha\eta\mathrm{cov}(\rho_i, \gamma), \quad E(s) \approx \alpha E(1 - \eta\gamma) \tag{18}$$

Insert (18) into (14) and (15), to obtain

$$E(\rho_i) = r + \frac{\eta\mathrm{cov}(\rho_i, \gamma)}{E(1 - \eta\gamma)} \tag{19}$$

and

$$E(\rho_M) = r + \frac{\eta\mathrm{cov}(\rho_M, \gamma)}{E(1 - \eta\gamma)} \tag{20}$$

and therefore

$$E(\rho_i) = r + [E(\rho_M) - r]\frac{\mathrm{cov}(\rho_i, \gamma)}{\mathrm{cov}(\rho_M, \gamma)} \tag{21}$$

(21) is equivalent to (11), by definition (10) of consumption beta.

(19) and (20) indicate that the spread between the expected rate of return and the pure rate of interest is the larger, the larger is the representative degree of relative risk aversion or the covariance between the market rate of return and the rate of growth of consumption. It is the smaller, the larger is the expected rate of growth of consumption. Looking at (20) in another way

$$\eta = \frac{E(\rho_M) - r}{E(\gamma) + \mathrm{cov}(\rho_M, \gamma)} \tag{22}$$

We can estimate the representative degree of relative risk aversion from the observations of market rate of return, pure rate of interest and rate of growth of consumption, using (22).

References

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