Dynamic Programming and Its Application to Households' Behaviour

Denzo Kamiya

26 May 1998

1 Dynamic programming problem in economics

1. The problem of dynamic programming that appears often in economics takes the following form: Find $\{x_t\}$ and $\{v_t\}$ so as to maximise

$$\begin{aligned} &\sum_{t=0}^{T-1} \alpha_t f(x_t, v_t, t) + S(x_T) \\ \text{subject to} \quad & x_{t+1} = g(x_t, v_t, t), t = 0, 1, 2, \cdots, T-1 \end{aligned}$$

given f, g, S, $\{\alpha_t\}$ and x_0.

2. For such a problem, Bellman's principle of optimality holds:

> *The principle of optimality*: An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision. [Bellman (1957), p.83.]

This principle justifies solving the problem in the reverse order, starting from the end and going back to the beginning of the planning period.

3. The maximised present value of the objective is a function of x representing the initial state. Name it the value function. The value function of the last period problem is

$$V(x, T) = S(x)$$

This maximisation problem is trivial, nothing being left unknown given the initial condition, x.

4. The second from the last period problem turns out to be to find x_T and v_{T-1} to maximise

$$\begin{aligned} &\alpha_{T-1} f(x_{T-1}, v_{T-1}, T-1) + V(x_T, T) \\ \text{subject to} \quad & x_T = g(x_{T-1}, v_{T-1}, T-1) \end{aligned}$$

given the initial condition $x_{T-1} = x$. The value function of period $T-1$ will be

$$\begin{aligned} V(x, T-1) &= \max[\alpha_{T-1} f(x, v_{T-1}, T-1) + V(x_T, T)] \\ &= \alpha_{T-1} f(x, v^*_{T-1}, T-1) + V(x^*_T, T) \end{aligned}$$

The asterisk indicates the maximising values.

5. At an arbitrary stage t between 0 and T, the problem is to find x_{t+1} and v_t to maximise

$$\begin{aligned} &\alpha_t f(x_t, v_t, t) + V(x_{t+1}, t+1) \\ &\text{subject to} \quad x_{t+1} = g(x_t, v_t, t) \end{aligned}$$

given the initial condition $x_t = x$. The value function of period t will be

$$V(x, t) = \alpha_t f(x, v^*_t, t) + V(x^*_{t+1}, t+1)$$

6. From the successive definitions of value functions, it is clear that the value function of period t has another expression

$$V(x, t) = \alpha_t f(x, v^*_t, t) + \sum_{s=t+1}^{T-1} \alpha_s f(x^*_s, v^*_s, s) + S(x^*_T)$$

That is the maximum value of the objective for period t

$$\sum_{s=t}^{T-1} \alpha_s f(x_s, v_s, s) + S(x_T)$$

with respect to $\{x_s\}$ and $\{v_s\}$ subject to

$$x_{s+1} = g(x_s, v_s, s), \quad s = t, t+1, t+2, \cdots, T-1$$

given the initial condition $x_t = x$.

7. The maximum solution for the two period problem in paragraph 5 must satisfy two conditions: the maximum condition,

$$\alpha_t f_v(x, v_t, t) + V_x(x_{t+1}, t+1) g_v(x, v_t, t) = 0$$

and the tangency condition following from the envelope theorem,

$$V_x(x, t) = \alpha_t f_x(x, v_t, t) + V_x(x_{t+1}, t+1) g_x(x, v_t, t)$$

or, inserting the maximum condition,

$$\frac{V_x(x, t) - \alpha_t f_x(x, v_t, t)}{g_x(x, v_t, t)} = -\frac{\alpha_t f_v(x, v_t, t)}{g_v(x, v_t, t)}$$

See Samuelson (1947), p. 34, for the envelope theorem.

8. The two period maximisation problem may be written in a simpler form using the current value function:

$$W(x,t) = \frac{V(x,t)}{\alpha_t}$$

Two period relation is

$$W(x,t) = f(x, v_t^*, t) + \beta_{t+1} W(x_{t+1}^*, t+1)$$
$$\beta_{t+1} = \frac{\alpha_{t+1}}{\alpha_t}$$

The problem now is to find x_{t+1} and v_t to maximise

$$f(x_t, v_t, t) + \beta_{t+1} W(x_{t+1}, t+1)$$
$$\text{subject to} \quad x_{t+1} = g(x_t, v_t, t)$$

given the initial condition $x_t = x$. The two conditions that the solution must satisfy are

$$f_v(x, v_t, t) + \beta_{t+1} W_x(x_{t+1}, t+1) g_v(x, v_t, t) = 0$$
$$W_x(x,t) = f_x(x, v_t, t) + \beta_{t+1} W_x(x_{t+1}, t+1) g_x(x, v_t, t)$$

or, using the maximum condition

$$\frac{W_x(x,t) - f_x(x, v_t, t)}{g_x(x, v_t, t)} = -\frac{f_v(x, v_t, t)}{g_v(x, v_t, t)}$$

9. Stochastic problem arises when function $g(x_t, v_t, t)$ defining the constraint is a stochastic function of v_t. Consider in this case maximisation of the mathematical expectation of the objective. The problem now is to find $\{x_t\}$ and $\{v_t\}$ to maximise

$$E\left[\sum_{t=0}^{T-1} \alpha_t f(x_t, v_t, t+1) + S(x_T)\right]$$
$$\text{subject to} \quad x_{t+1} = g(x_t, v_t, t), t = 0, 1, 2, \cdots, T-1$$

given the initial condition x_0. The solutions for $\{x_t\}$ and $\{v_t\}$ are stochastic processes.

10. The two period problem is to find x_{t+1} and v_t to maximise

$$f(x_t, v_t, t) + \beta_{t+1} E[W(x_{t+1}, t+1)]$$
$$\text{subject to} \quad x_{t+1} = g(x_t, v_t, t)$$

given the initial condition $x_t = x$.

Solve the two period problem successively in the reverse order, taking x as non-stochastically given at each stage. The conditions that the maximum solution must satisfy are

$$f_v(x, v_t, t) + \beta_{t+1} E[W_x(x_{t+1}, t+1) g_v(x, v_t, t)] = 0$$

$$W_x(x,t) = f_x(x, v_t, t) + \beta_{t+1} E[W_x(x_{t+1}, t+1) g_x(x, v_t, t)]$$

v_t is a non stochastic function of the realised value of x, and is stochastic because x_t is stochastic.

11. Considered from the economic point of view, the individual knows what has happened up to the time of decision making. That is, in period t, he knows the value of $x_s, s = t, t-1, t-2, \cdots$. Given that information, the individual maximises the expected utility of the future consumption, under the stochastic budget constraint. Of the values of $\{x_t\}$ that he knows of the past, only x_t matters for the decision of period t. The individual determines v_t depending on the realised value of x_t.

12. If function $f(x, v_t, t)$ or $S(x)$ is stochastic and $g(x_t, v_t, t)$ is not, we have also to consider maximisation of the mathematical expectation. Despite this apparent similarity to the stochastic problem, however, the problem is essentially non-stochastic, the solution being non-stochastic.

2 The Theory of Lifetime Portfolio Selection

1. Consider a household earning $y_t, t = 0, 1, 2, \cdots$ of labour income and planning consumption and asset holdings for T periods. It can hold riskless asset with one period rates of return r_t, $t = 0, 1, 2, \cdots$, or risky asset with one period rates of return z_t, $t = 0, 1, 2, \cdots$. We suppose y_t and z_t to be random. The labour income and the rates of return are determined in the market, and the household behaves as price-taker. Its objective is to maximise the expected utility of future consumptions and of the wealth at the end of planning period.

2. The household's planning problem is one of stochastic dynamic programming. That is, to find $\{x_t\}$, $\{a_t\}$ and $\{c_t\}$ so as to maximise

$$E\left[\sum_{t=0}^{T-1} \alpha_t u(c_t) + S(x_T)\right]$$

subject to

$$\begin{gathered} x_{t+1} = [(1 + r_t) + (z_t - r_t)a_t](x_t + y_t - c_t) \\ t = 0, 1, 2, \cdots, T-1 \end{gathered}$$

given $\{\alpha_t\}, \{y_t\}$ and the initial condition x_0.

3. The two period problem is to find x_{t+1}, c_t and a_t to maximise

$$\begin{aligned} &u(c_t) + \beta_{t+1} E[W(x_{t+1}, t+1) \\ \text{subject to} \quad &x_{t+1} = [(1 + r_t) + (z_t - r_t)a_t](x_t + y_t - c_t) \end{aligned}$$

given y_t and the initial condition $x_t = x$.

The solution must satisfy

$$\begin{aligned} E[W_x(x_{t+1}, t+1)g_a(x, y_t, c_t, a_t, t)] &= 0 \\ u_c(c_t) + \beta_{t+1} E[W_x(x_{t+1}, t+1)g_c(x, y_t, c_t, a_t, t)] &= 0 \\ W_x(x, t) &= \beta_{t+1} E[W_x(x_{t+1}, t+1)g_x(x, y_t, c_t, a_t, t)] \end{aligned}$$

From the definition of the constraint, it is obvious that

$$g_a(x, y_t, c_t, a_t, t) = (z_t - r_t)[x + y_t - c_t]$$
$$-g_c(x, y_t, c_t, a_t, t) = g_x(x, y_t, c_t, t) = (1 + r_t) + (z_t - r_t)a_t$$

Inserting these into the conditions for the maximum solution, we have

$$E[W_x(x_{t+1}, t+1)(z_t - r_t)] = 0$$
$$u_c(c_t) - \beta_{t+1}E[W_x(x_{t+1}, t+1)((1 + r_t) + (z_t - r_t)a_t)] = 0$$
$$W_x(x, t) = \beta_{t+1}E[W_x(x_{t+1}, t+1)((1 + r_t) + (z_t - r_t)a_t)]$$

From the last two equalities, we have

$$W_x(x, t) = u_c(c_t)$$

and by analogy

$$W_x(x_{t+1}, t+1) = u_c(c_{t+1})$$

Inserting this equality into the first two of the conditions for the maximum solution, and rearranging terms, we have

$$E[(z_t - r_t)u_c(c_{t+1})] = 0$$
$$u_c(c_t) - (1 + r_t)\beta_t E[u_c(c_{t+1})] - a_t\beta_t E[(z_t - r_t)u_c(c_{t+1})] = 0$$

Therefore, we finally have

$$\frac{u_c(c_t)}{\beta_t E[u_c(c_{t+1})]} = 1 + r_t$$

4. This relation may be viewed also as

$$E[u_c(c_{t+1})] = \frac{1 + \rho_t}{1 + r_t}u_c(c_t), \quad 1 + \rho_t = \frac{1}{\beta_t}$$

or

$$u_c(c_{t+1}) = \frac{1 + \rho_t}{1 + r_t}u_c(c_t) + \epsilon_t$$

and if $\rho_t = r_t$,

$$u_c(c_{t+1}) = u_c(c_t) + \epsilon_t$$

5. Samuelson (1969) showed that if the household has no labour income and no bequest, and if the utility function is isoelastic with positive elasticity less than one, that is, the degree of relative risk aversion is a positive constant less than one:

$$u(c) = c^{1-\sigma}$$

then, the value function also is isoelastic, the elasticity being equal to that of the utility function:

$$W(x, t) = A_t x^{1-\sigma}$$

References

Lectures. Chapter 6.

Richard Bellman (1957) *Dynamic Programming.* Pinceton, New Jewsey: Princeton University Press.

Paul A. Samuelson (1969) "Life Time Portfolio Selection by Stochastic Dynamic Programming." *Review of Economics and Statistics* 51: 239 – 246.

Paul A. Samuelson (1947) *Foundations of Economic Analysis.* Cambridge, Massachusetts: Harvard University Press. Chapter III, pp. 34 – 36. See also pp. 36 – 39.

Robert E. Hall (1978) "Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence." *Journal of Political Economy* 86: 971 – 987.

付録：　包絡線定理

定理：　最大化の目的関数

$$y = f(x_1, x_2, z)$$

について，z を所与としたときに y を最大化する x_1，x_2 の値 を $x_1^*(z)$，$x_2^*(z)$，y の最大値を $v(z)$ としよう．そのとき，次の関係が成り立つ．

$$\frac{d}{dz}v(z) = f_z(x_1^*, x_2^*, z)$$

ここで

$$f_z(x_1, x_2, z) = \frac{\partial}{\partial z}f(x_1, x_2, z)$$

証明：　$v(z)$ の定義から，

$$v(z) = f(x_1^*, x_2^*, z)$$

したがって

$$\frac{d}{dz}v(z) = f_1(x_1^*, x_2^*, z)\frac{dx_1^*}{dz} + f_2(x_1^*, x_2^*, z)\frac{dx_2^*}{dz} + f_z(x_1^*, x_2^*, z)$$

ここで

$$f_1(x_1, x_2, z) = \frac{\partial}{\partial x_1}f(x_1, x_2, z), \quad f_2(x_1, x_2, z) = \frac{\partial}{\partial x_2}f(x_1, x_2, z)$$

一方，x_1^*，x_2^* は z を所与として $f(x_1, x_2, z)$ を最大化する x_1，x_2 の値であるから

$$f_1(x_1^*, x_2^*, z) = 0, \quad f_2(x_1^*, x_2^*, z) = 0$$