Introduction to Finance (PCP)

Solutions for Exercise 3

1. 1-3. From the view of the 1st and 2nd FTAPs, it suffices to see that the equation $D\mathbf{q} = \mathbf{S}_0$ has a unique solution which is positive, where $\mathbf{q} = (q_1, q_2, q_3)^\top$. By simple calculation, we have $\mathbf{q} = (1/3, 1/3, 1/3)^\top$ is the unique solution to the above equation. Thus, the market is arbitrage-free and complete. The martingale probability $\mathbb{Q}$ is given by $\mathbb{Q}(\{\omega_j\}) = 1/3$ for any $j = 1, 2, 3$.
4. Noting that the replicating portfolio $\mathbf{x}$ is given as the solution to the equation $\mathbf{x}D = \mathbf{C}$, we have $\mathbf{x} = (-10/9, 0, 1/9)$. On the other hand, the price of $\mathbf{C}$, denoted by C_0, is given as $C_0 = \mathbf{x} \cdot \mathbf{S}_0$ or $C_0 = \mathbb{E}_{\mathbb{Q}}[\mathbf{C}] = \mathbf{q} \cdot \mathbf{C}$, from which we obtain $C_0 = 10/3$.

2. 1-2. Note that $\widehat{D} = \frac{1}{1.1}D$. Let $\mathbf{q} = (q_1, q_2, q_3)^\top$. Solving $\widehat{D}\mathbf{q} = \mathbf{S}_0$, we can see that $\mathbf{q} = \left(\frac{2}{3}(1-a), a, \frac{1}{3}(1-a)\right)^\top$ is a solution for any $a \in (0, 1)$. Thus, the equation $\widehat{D}\mathbf{q} = \mathbf{S}_0$ has infinitely many positive solutions $\mathbf{q}$. In other words, the market is arbitrage-free, but not complete.

3. Letting $\mathbf{q} = (q_1, q_2, q_3, q_4)^\top$ be a solution to the equation $D\mathbf{q} = \mathbf{S}_0$, we have $q_1 + q_2 = 1/2$ and $q_3 + q_4 = 1/2$ for any $X > 0$.
1. When $X = 12$, $\mathbf{q}$ is a solution to $D\mathbf{q} = \mathbf{S}_0$ if and only if $q_1 + q_2 = 1/2$, $q_3 + q_4 = 1/2$ and $q_1 + 3q_3 = 1/2$ hold. Thus, there are infinitely many positive solutions. As a result, the market is arbitrage-free, but not complete.
2. When $X = 11$, $q_1 + 2q_3 = 0$ holds. Thus, there is no positive solution. Then the no-arbitrage condition does not hold.
Let $\mathbf{x} = (x_1, x_2, x_3)$ be an arbitrage portfolio. Thus, $\mathbf{x} \cdot \mathbf{S}_0 = 0$ holds, and $\mathbf{x}D$ is a nonnegative vector having at least one positive entry. Now, we have $\mathbf{x}D = (-2x_2 - 2x_3, -x_2 - 2x_3, -x_2 + 2x_3, x_2 + 2x_3)^\top$. Since $\mathbf{x}D$ is nonnegative, $x_2 + 2x_3 = 0$ holds. Thus, we have $\mathbf{x}D = (2x_3, 0, 4x_3, 0)^\top$, which implies $x_3 > 0$ follows. For example, $\mathbf{x} = (8, -2, 1)$ is an example of arbitrage portfolios.
3. Define $g(\mathbf{y}) := D\mathbf{y}$, where $\mathbf{y} = (y_1, y_2, y_3, y_4)^\top$. By the proof of the 2nd FTAP, the market is complete if and only if $\mathrm{Ker}(g) = \{\mathbf{0}\}$, where $\mathrm{Ker}(g) := \{\mathbf{y} | g(\mathbf{y}) = \mathbf{0}\}$. Now, we have $\mathbf{y} \in \mathrm{Ker}(g) \iff D\mathbf{y} = \mathbf{0} \iff$ "$y_1 + y_2 = 0$, $y_3 + y_4 = 0$ and $y_2 + (X - 9)y_4 = 0$". Thus, we obtain $\mathrm{Ker}(g) = \{((X-9)a, -(X-9)a, -a, a)^\top | a \in \mathbb{R}\}$, from which the market is not complete for any $X > 0$.