Introduction to Finance (PCP)

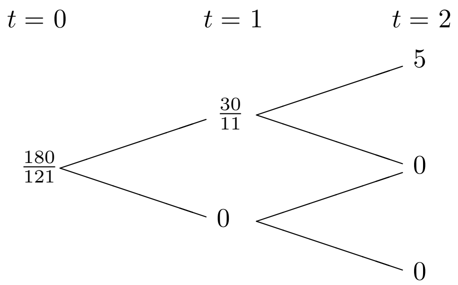
Solutions for Exercise 2

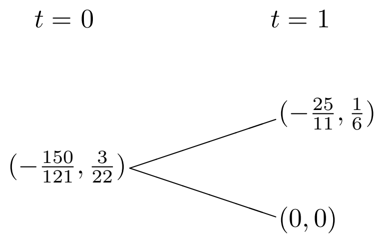
NOTE:

1) S_t denotes the price of the risky asset at time t.

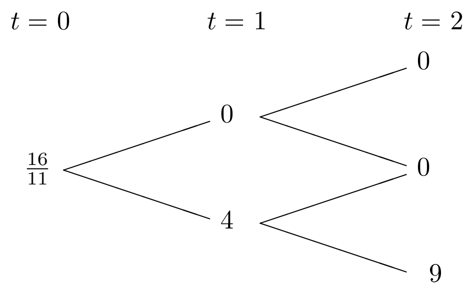
2) $\mathbb{Q}$ denotes a martingale probability.

1. 1. $\mathbb{Q}(S_1 = 30) = \frac{3}{5}$, $\mathbb{Q}(S_1 = 10) = \frac{2}{5}$;
$\mathbb{Q}(S_2 = 45) = \frac{9}{25}$, $\mathbb{Q}(S_2 = 15) = \frac{12}{25}$, $\mathbb{Q}(S_2 = 5) = \frac{4}{25}$,
2. The left and right figures give the price process and the replication portfolio, respectively. In the right figure, (a, b) represents the portfolio composed of a shares of the riskless asset and b shares of the risky assets.



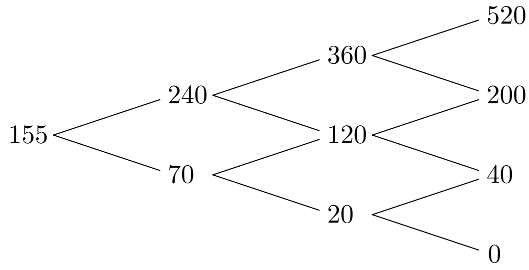


3. The price process is given in the following figure, and early exercise is implemented at the node $S_1 = 10$.

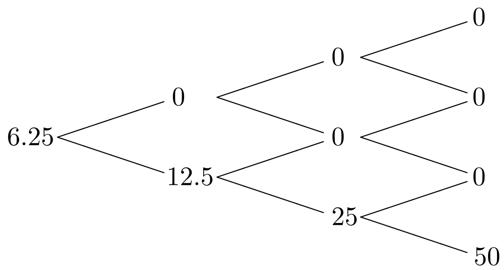


2. 1. $\mathbb{Q}(S_3 = 320) = \frac{3}{8}$,

2.



3.



4. Early exercise never occur.

3. We consider a T-period binomial model with short rate $r \geq 0$. Let $(S_t)_{t=0,\ldots,T}$ be the risky asset price process such that S_{t+1}/S_t takes the value of either u or d for any $t = 0, \ldots, T-1$, where $u > 1 + r > d > 0$. Moreover, q denotes $\mathbb{Q}(S_1 = uS_0)$. Then, C_t the price of the call option with strike price K at time t is given by

$$C_t = (1+r)^{-(T-t)} \sum_{k=m}^{T-t} \binom{T-t}{k} q^k (1-q)^{T-t-k} (u^k d^{T-t-k} S_t - K),$$

where m is the least integer satisfying $u^k d^{T-t-k} S_t \geq K$. Note that C_t is given as a function of S_t.

4. To see the first inequality, we suppose that

$$C_0 - P_0 - S_0 + K(1+r)^{-T} > 0 \qquad (1)$$

holds. It suffices to find an arbitrage portfolio under (1). Now, we construct the following portfolio:

	$t=0$		$t=s$		$t=T$	
Call	Selling	C_0	Exercise[1]	$-(S_s - K)$		
Put	Buying	$-P_0$			Exercise	$(K - S_T)^+$
Risky asset	Buying	$-S_0$	Selling	S_s		
Riskless asset	Borrowing[2]	$-C_0 + P_0 + S_0$	Lending	$-K$	See below	
Total	0		0			> 0

At the maturity, the investor receives $(C_0 - P_0 - S_0)(1+r)^T + K(1+r)^{T-s}$ shares of the riskless asset. (1) implies that this amount is greater than $-K + K(1+r)^{T-s}$, which is nonnegative. As a result, the above portfolio is an arbitrage. This contradicts to the no-arbitrage condition. We can conclude that (1) does not hold.

Next, we prove the second inequality by the same sort argument as the above. Supposing that

$$-C_0 + P_0 + S_0 - K > 0 \qquad (2)$$

holds, we construct the following portfolio:

	$t=0$		$t=s$		$t=T$	
Call	Buying	$-C_0$			Exercise	$(S_T - K)^+$
Put	Selling	P_0	Exercise[3]	$-(K - S_s)$		
Risky asset	Selling	S_0	Clearing	$-S_s$		
Riskless asset	Lending	$C_0 - P_0 - S_0$	Borrowing	K	See below	
Total	0		0			> 0

At the maturity, the investor receives $(-C_0+P_0+S_0)(1+r)^T - K(1+r)^{T-s}$ shares of the riskless asset. (2) implies that this amount is greater than $K(1+r)^T - K(1+r)^{T-s} \geq 0$. Thus, the above portfolio is an arbitrage. This contradicts to the no-arbitrage condition. We can conclude that (2) does not hold.

[1] We suppose that the holder of the call option exercises the option at time s. Without loss of generality, we may assume $S_s - K \geq 0$. If the call option is not exercised at all, then the investor sells the risky asset she holds at the maturity. In this case, her cashflow at the maturity is given as $(K - S_T)^+ + S_T + (C_0 - P_0 - S_0)(1+r)^T > (K - S_T)^+ + S_T - K \geq 0$.

[2] If $-C_0 + P_0 + S_0$ is negative, the investor lends $C_0 - P_0 - S_0$ at $t = 0$.

[3] We suppose that the holder of the put option exercises the option at time s. Without loss of generality, we may assume $K - S_s \geq 0$. If the put option is not exercised at all, then the investor settles her position of the risky asset at the maturity. In this case, her cashflow at the maturity is given as $(S_T-K)^+ - S_T + (-C_0+P_0+S_0)(1+r)^T > (S_T-K)^+ - S_T + K(1+r)^T \geq 0$.

5. Let K be the strike price. Since the function $f(x) = (K - x)^+$ is convex, Jensen's inequality implies

$$\mathbb{E}_{\mathbb{Q}}[(K - S_{t+1})^+|S_t] \geq (K - \mathbb{E}_{\mathbb{Q}}[S_{t+1}|S_t])^+ = (K - S_t)^+$$

for any $t = 0, \ldots, T-1$, where $\mathbb{E}_{\mathbb{Q}}$ means the expectation under $\mathbb{Q}$. Thus, early exercise never occur.